LINEAR INTEGRAL EQUATIONS

BY

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PREFACE

For many years the subject of functional equations has held a prominent place in the attention of mathematicians. In more recent years this attention has been directed to a particular kind of functional equation, an integral equation, wherein the unknown function occurs under the integral sign. The study of this kind of equation is sometimes referred to as the inversion of a definite integral.

In the present volume I have tried to present in a readable and systematic manner the general theory of linear integral equations with some of its applications. The applications given are to differential equations, calculus of variations, and some problems in mathematical physics. The applications to mathematical physics herein given are to Neumann's problem, Dirichlet's problem, and certain vibration problems which lead to differential equations with boundary conditions. The attempt has been made to present the subject matter in such a way as to make the volume available as a text on this subject in Colleges and Universities.

The reader who so desires can omit the chapters on the applications. The remaining chapters on the general theory are an entity in themselves.

The discussion has been confined to those equations which are linear and in which a single integration occurs. The limits of the present volume forbid any adequate treatment of integral equations in several independent variables; systems of integral equations; integral equations of higher order; integro-differential equations; singular integral equations; integral equations with special or discontinuous kernels.

I desire here to express my thanks to Prof. Oscar Bolza (now of Freiburg University, formerly of the University of Chicago) for his permission to make use of my notes on his lectures on integral equations delivered during the summer of 1913 at the University of Chicago. A bound volume of these notes in my handwriting has resided these past ten years on the shelves of the University of Chicago mathematical library and has been available to many students during this time. A number of copies of these notes are in circulation in this country at present.

The following books have been available and have been found to be of value in the preparation of this volume:

M. Bocher: An Introduction to the Study of Integral Equations. No. 10, Cambridge Tracts, 1909. University Press. Cambridge.

É. Goursat: Cours D'Analyse Mathématique. Tome III. Chaps. 30, 31, 32, 33. Paris, Gauthier-Villars, 1923.

Heywood-Fréchet: L'Équation de Fredholm et ses applications à la Physique Mathématique. Paris, Hermann et Fils, 1912.

Kneser: Die Integralgleichungen und ihre Anwendungen in der Math. Physik. Braunschweig. Vieweg et Sohn, 1922.

G. Kowalewski: Einführung in die Determinanten Theorie. 18 and 19 cap. (p. 455–505). Verlag von Veit et C., 1909.

Lalesco: Introduction à la Théorie des Équations Intégrales. Paris. Hermann et Fils, 1912.

Volterra: Leçons sur les Équations Intégrales et les Équations Intégro-Differentielles. Paris, Gautheir-Villars, 1913.

For those who desire a bibliography on this subject we refer the reader to a short bibliography in the work by Heywood-Fréchet and to a more extensive bibliography in the work by Lalesco.

W. V. Lovitt.

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LINEAR INTEGRAL EQUATIONS

CHAPTER I

INTRODUCTORY

1. Linear Integral Equation of the First Kind.—An equation of the form

$$(1) \qquad \int_a^b K(x, t)u(t)dt = f(x)$$

is said to be a **linear integral equation of the first kind.** The functions $K(x, t)$ and $f(x)$ and the limits a and b are known. It is proposed so to determine the unknown function u that (1) is satisfied for all values of x in the closed interval $a \leq x \leq b$. $K(x, t)$ is called the *kernel* of this equation.

Instead of equation (1), we have often to deal with equations of exactly the same form in which the upper limit of integration is the variable x. Such an equation is seen to be a special case of (1) in which the kernel $K(x, t)$ vanishes when $t > x$, since it then makes no difference whether x or b is used as the upper limit of integration.

The characteristic feature of this equation is that the unknown function u occurs under a definite integral. Hence equation (1) is called an *integral* equation and, since u occurs linearly, equation (1) is called a *linear* integral equation.

2. Abel's Problem.—As an illustration of the way in which integral equations arise, we give here a statement of Abel's problem.

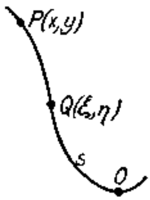


FIG. 1.

Given a smooth curve situated in a vertical plane. A particle starts from rest at any point P. Let us find, under the action of gravity, the time T of descent to the lowest point O. Choose O as the origin of coordinates, the x-axis vertically upward, and the y-axis horizontal. Let the coordinates of P be (x, y), of Q be (ξ, η), and s the arc OQ.

The velocity of the particle at Q is

$$\frac{ds}{dt} = -\sqrt{2g(x - \xi)}.$$

Hence

$$t = -\int_P^Q \frac{ds}{\sqrt{2g(x - \xi)}}.$$

The whole time of descent is, then,

$$T = \int_o^P \frac{ds}{\sqrt{2g(x - \xi)}}.$$

If the shape of the curve is given, then s can be expressed in terms of ξ and hence ds can be expressed in terms of ξ. Let

$$ds = u(\xi)d\xi.$$

Then

$$T = \int_o^x \frac{u(\xi)d\xi}{\sqrt{2g(x - \xi)}}.$$

Abel set himself the problem[1] of finding that curve for which the time T of descent is a given function of x, say $f(x)$.

[1] For a solution of this problem, see BÔCHER, "Integral Equations," p. 8, Cambridge University Press, 1909.

Our problem, then, is to find the unknown function u from the equation

$$f(x) = \int_0^x \frac{1}{\sqrt{2g(x-\xi)}} u(\xi)d\xi.$$

This is a linear integral equation of the first kind for the determination of u.

3. Linear Integral Equation of the Second Kind.—An equation of the form

$$u(x) = f(x) + \int_a^b K(x, t)u(t)dt \tag{2}$$

is said to be a **linear integral equation of the second kind.**

$K(x, t)$ is called the kernel of this equation. The functions $K(x, t)$ and $f(x)$ and the limits a and b are known. The function u is unknown.

The equation

$$u(x) = f(x) + \int_a^x K(x, t)u(t)dt$$

is known as Volterra's linear integral equation of the second kind.

If $f(x) \equiv 0$, then

$$u(x) = \int_a^b K(x, t)u(t)dt.$$

This equation is said to be a homogeneous linear integral equation of the second kind.

Sometimes, in order to facilitate the discussion, a parameter λ is introduced, thus

$$u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt.$$

This equation is said to be a linear integral equation of the second kind with a parameter.

Linear integral equations of the first and second kinds are special cases of the linear integral equation of the third kind:

$$\Psi(x)u(x) = f(x) + \int_a^b K(x, t)u(t)dt.$$

Equation (1) is obtained if $\Psi(x) \equiv 0$.
Equation (2) is obtained if $\Psi(x) \equiv 1$.

4. Relation between Linear Differential Equations and Volterra's Integral Equation.—Consider the equation

$$(3) \qquad \frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ . \ . \ . + a_n(x)y = \varphi(x),$$

where the origin is a regular point for the $a_i(x)$.

Let us make the transformation

$$\frac{d^n y}{dx^n} = u(x).$$

Then

$$(4) \left\{ \begin{aligned} \frac{d^{n-1}y}{dx^{n-1}} &= \int_o^x u(x)dx + C_1 \\ &. \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ y &= \int_o^x u(x)dx^n + C_1\frac{x^{n-1}}{\lfloor n-1} + C_2\frac{x^{n-2}}{\lfloor n-2} + \ . \ . \ . + C_n, \end{aligned} \right.$$

where $\int_o^x u(x)dx^n$ stands for a multiple integral of order n.

Equations (4) transform (3) into

$$(5) \ u(x) + a_1(x)\int_o^x u(x)dx + \ . \ . \ . + a_n(x)\int_o^x u(x)dx^n = \varphi(x) + \sum_1^n C_i\alpha_i(x),$$

where

$$\alpha_i(x) = a_i(x) + \frac{x}{1}a_{i+1}(x) + \ . \ . \ . + a_n(x)\frac{x^{n-i}}{\lfloor n-i}.$$

If we now put

$$\varphi(x) + \sum_{1}^{n} C_i\alpha_i(x) = f(x),$$

and make use of the well-known formula

$$\int_o^x u(t)dt^n = \int_o^x \frac{(x-t)^{n-1}}{\underline{|n-1}}\, u(t)dt,$$

equation (5) becomes

$$u(x) + \int_o^x \Bigg[a_1(x) + a_2(x)(x-t) + \cdot \ \cdot \ \cdot + a_n(x)\, \frac{(x-t)^{n-1}}{\underline{|n-1}} \Bigg] u(t)dt = f(x),$$

which is a Volterra integral equation of the second kind.

In order that the right-hand member of (5) have a definite value it is necessary that the coefficients C_i have definite values. Then, inversely, the solution of the Volterra's equation (5) is equivalent to the solution of Cauchy's problem for the linear differential equation (3). The uniqueness of the solution of Volterra's equation follows from the fact that Cauchy's problem admits for a regular point one and only one solution.[1]

5. Non-linear Equations.—This work will be confined to a discussion of linear integral equations. It is desirable, however, at this point to call the reader's attention to some integral equations which are non-linear.

The unknown function may appear in the equation to a power n greater than 1, for example;

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u^n(t)dt.$$

The unknown function may appear in a more general way, as indicated by the following equation:

$$u(x) = f(x) + \lambda \int_a^b \varphi[x, t, u(t)]dt.$$

[1] For further discussion consult Lalesco, T., "Théorie Des Équations Intégrales," pp. 12ff, Herman and Fils, Paris, 1912.

In particular, the differential equation

$$\frac{du}{dx} = \varphi(x, u)$$

can be put in the integral form

$$u(x) = C + \int_o^x \varphi[t, u(t)]dt.$$

Still other general types of non-linear integral equations have been considered. Studies have also been made of systems of integral equations both linear and non-linear. Some study has been made of integral equations in more than one variable, for example;

$$u(x, y) = f(x, y) + \lambda \int_a^b \int_c^d K(x, y; t_1, t_2)u(t_1, t_2)dt_1dt_2.$$

6. Singular Equations.—An integral equation is said to be singular when either one or both of the limits of integration become infinite, for example;

$$u(x) = f(x) + \lambda \int_o^\infty \sin(xt)u(t)dt.$$

An integral equation is also said to be singular if the kernel becomes infinite for one or more points of the interval under discussion, for example;

$$f(x) = \int_o^x \frac{H(x, t)}{(x - t)^\alpha}u(t)dt \qquad (0 < \alpha < 1).$$

Abel's problem, as stated in §2, is of this character. Abel set himself the problem of solving the more general equation

$$f(x) = \int_a^x \frac{u(t)dt}{(x - t)^\alpha} \qquad (0 < \alpha < 1).$$

7. Types of Solutions.—By the use of distinct methods, the solution of a linear integral equation of the second kind with a parameter λ has been obtained in three different forms:

1. The first method, that of successive substitutions, due to Neumann, Liouville, and Volterra, gives us $u(x)$ as an integral series in λ, the coefficients of the various powers of λ being functions of x. The series converges for values of λ less in absolute value than a certain fixed number.

2. The second method, due to Fredholm, gives $u(x)$ as the ratio of two integral series in λ. Each series has an *infinite* radius of convergence. In the numerator the coefficients of the various powers of λ are functions of x. The denominator is independent of x. For those values of λ for which the denominator vanishes, there is, in general, no solution, but the method gives the solution in those exceptional cases in which a solution does exist. The solution is obtained by regarding the integral equation as the limiting form of a system of n linear algebraic equations in n variables as n becomes infinite.

3. The third method, developed by Hilbert and Schmidt, gives $u(x)$ in terms of a set of *fundamental functions*. The functions are, in the ordinary case, the solutions of the corresponding homogeneous equation

$$u(x) = \lambda \int_a^b K(x, t)u(t)dt.$$

In general, this equation has but one solution:

$$u(x) \equiv 0.$$

But there exists a set of numbers,

$$\lambda_1, \lambda_2, \quad . \quad . \quad . \quad , \lambda_n, \quad . \quad . \quad . \quad ;$$

called *characteristic constants* or *fundamental numbers*, for each of which this equation has a finite solution:

$$u_1(x), u_2(x), \quad . \quad . \quad . \quad , u_n(x), \quad . \quad . \quad .$$

These are the fundamental functions. The solution then is obtained in the form

$$u(x) = \sum C_n u_n(x),$$

where the C_n are arbitrary constants.

EXERCISES

Form the integral equations corresponding to the following differential equations with the given initial conditions:

1. $\dfrac{d^2y}{dx^2} + y = 0,\ x = 0,\ y = 0,\ y' = 1,\ y'' = 0.$

$$Ans.\ u(x) = x + \int_o^x (t - x)u(t)dt.$$

2. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0,\ x = y = 0,\ y' = -1,\ y'' = -5.$

$$Ans.\ u(x) = 29 + 6x + \int_o^x (6x - 6t + 5)u(t)dt.$$

3. $\dfrac{d^2y}{dx^2} + y = \cos x,\ x = y = 0,\ y' = 1,\ y'' = 2.$

$$Ans.\ u(x) = \cos x - x - 2 + \int_o^x (t - x)u(t)dt.$$

4. $\dfrac{dy}{dx} - y = 0,\ x = 0,\ y = y' = 1.$

$$Ans.\ u(x) = 1 + \int_o^x u(t)dt.$$

5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 5y = 0.$

$$Ans.\ u(x) = 1 - 2x - 4x^2 + \int_o^x [3 + 6(x - t) - 4(x - t)^2]u(t)dt.$$

CHAPTER II

SOLUTION OF INTEGRAL EQUATION OF SECOND KIND BY SUCCESSIVE SUBSTITUTIONS

8. Solution by Successive Substitutions.—We proceed now to a solution of the linear integral equation of the second kind with a parameter. We take up first the case where both limits of integration are fixed (*Fredholm's equation*). We assume that

(1) *a*) $u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$, ($a$, b, constants).

b) $K(x, t) \not\equiv 0$, is real and continuous in the rectangle R, for which $a \leqq x \leqq b$ and $a \leqq t \leqq b$.

c) $f(x) \not\equiv 0$, is real and continuous in the interval I, for which $a \leqq x \leqq b$.

d) λ, constant.

We see at once that if there exists a continuous solution $u(x)$ of (1) and $K(x, t)$ is continuous, then $f(x)$ must be continuous. Hence the inclusion of condition (c) above.

Substitute in the second member of (1), in place of $u(t)$, its value as given by the equation itself. We find

$$u(x) = f(x) + \lambda \int_a^b K(x, t)\left[f(t) + \lambda \int_a^b K(t, t_1)u(t_1)dt_1\right]dt$$

$$= f(x) + \lambda \int_a^b K(x, t)f(t)dt$$

$$+ \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1)u(t_1)dt_1dt.$$

Here again we substitute for $u(t_1)$ its value as given by (1).

We get

$$u(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) \Big[f(t_1) + \lambda \int_a^b K(t_1, t_2) u(t_2) dt_2 \Big] dt_1 dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \lambda^3 \int_a^b K(x,t) \int_a^b K(t, t_1) \int_a^b K(t_1, t_2) u(t_2) dt_2 dt_1 dt.$$

Proceeding in this way we obtain

(2) $$u(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \ldots + \lambda^n \int_a^b K(x, t) \int_a^b K(t, t_1) \ldots \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \ldots dt_1 dt + R_{n+1}(x),$$

where

$$R_{n+1}(x) = \lambda^{n+1} \int_a^b K(x, t) \int_a^b K(t, t_1) \ldots \int_a^b K(t_{n-1}, t_n) u(t_n) dt_n \ldots dt_1 \, dt.$$

This leads us to the consideration of the following infinite series:

(3) $$f(x) + \lambda \int K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \ldots$$

Under our hypotheses b) and c), each term of this series is continuous in I. This series then represents a continuous function in I, provided it converges uniformly in I.

Since $K(x, t)$ and $f(x)$ are continuous in R and I respectively, $|K|$ has a maximum value M in R and $|f(x)|$ has a maximum value N in I:

$$|K(x, t)| \leqq M \quad \text{in } R$$
$$|f(x)| \leqq N \quad \text{in } I.$$

Put $S_n(x) = \lambda^n \int_a^b K(x, t) \int_a^b K(t, t_1) \cdots \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \cdots dt_1 dt.$

Then $$|S_n(x)| \leq |\lambda^n| N M^n (b - a)^n.$$

The series of which this is a general term converges only when

$$|\lambda| M (b - a) < 1.$$

Thus we see that the series (3) converges absolutely and uniformly when

$$|\lambda| < \frac{1}{M(b - a)}.$$

If (1) has a continuous solution, it must be expressed by (2). If $u(x)$ is continuous in I, its absolute value has a maximum value U. Then

$$|R_{n+1}(x)| < |\lambda^{n+1}| U M^{n+1} (b - a)^{n+1}.$$

If $$|\lambda| M (b - a) < 1, \text{ then}$$
$$\lim_{n \to \infty} R_{n+1}(x) = 0.$$

Thus we see that the function $u(x)$ satisfying (2) is the continuous function given by the series (3).

We can verify by direct substitution that the function $u(x)$ defined by (3) satisfies (1) or, what amounts to the same thing, place the series given by (3) equal to $u(x)$, multiply both sides by $\lambda K(x, t)$ and integrate term by term,[1] as we have a right to do. We obtain

$$\lambda \int_a^b K(x, t)u(t)dt = \lambda \int_a^b K(x,t)\Big[f(t) + \lambda \int_a^b K(t, t_1)f(t_1)dt_1 + \cdots \Big]dt$$

$$= \lambda \int_a^b K(x, t)f(t)dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1)f(t_1)dt_1dt + \cdots$$

$$= u(x) - f(x).$$

Thus we obtain the following:

Theorem I.—If

a) $u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$ (a, b, constants).

b) $K(x, t)$ is real and continuous in a rectangle R, for which $a \leqq x \leqq b$, $a \leqq t \leqq b$. $|K(x, t)| \leq M$ in R, $K(x, t) \not\equiv 0$.

c) $f(x) \not\equiv 0$, is real and continuous in $I: a \leqq x \leqq b$.

d) λ constant, $|\lambda| < \dfrac{1}{M(b - a)}$,

then the equation (1) has one and only one continuous solution in I and this solution is given by the absolutely and uniformly convergent series (3).

The equation

$$u(x) = f(x) + \int_a^b K(x, t)u(t)dt \tag{4}$$

is a special case of the equation (1), for which $\lambda = 1$. The discussion just made holds without change after putting $\lambda = 1$.

[1] Goursat-Hedrick, "Mathematical Analysis, vol. 1, §174, Ginn & Co.

Equations (1) and (4) may have a continuous solution, even though the hypothesis *d*),

$$|\lambda| M(b - a) < 1,$$

is not fulfilled. The truth of this statement is shown by the following example:

$$u(x) = \frac{x}{2} - \frac{1}{3} + \int_0^1 (x + t)u(t)dt,$$

which has the continuous solution $u(x) = x$, while

$$|\lambda| M(b - a) = 2 \nless 1.$$

9. Volterra's Equation.—The equation

$$(5) \qquad u(x) = f(x) + \lambda \int_a^x K(x, t)u(t)dt$$

is known as *Volterra's equation.*

Let us substitute successively for $u(t)$ its value as given by (5). We find

$$(6) \quad u(x) = f(x) + \lambda \int_a^x K(x, t)f(t)dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1)f(t_1)dt_1\, dt + \cdots + \lambda^n \int_a^x K(x, t) \int_a^t K(t, t_1) \cdots \int_a^{t_{n-2}} K(t_{n-2}, t_{n-1})f(t_{n-1})dt_{n-1} \cdots dt_1\, dt + R_{n+1}(x),$$

where

$$R_{n+1}(x) = \lambda^{n+1} \int_a^x K(x, t) \int_a^t K(t, t_1) \cdots \int_a^{t_{n-1}} K(t_{n-1}, t_n)\, u(t_n)dt_n \cdots dt_1\, dt.$$

We consider the infinite series.

$$(6') \quad u(x) = f(x) + \lambda \int_a^x K(x, t)f(t)dt + \lambda^2 \int_a^x K(x, t) \int_a^t K(t, t_1)f(t_1)dt_1\, dt + \cdots$$

The general term $V_n(x)$ of this series may be written

$$V_n(x) = \lambda^n \int_a^x K(x, t) \int_a^t K(t, t_1) \quad . \quad . \quad . \int_a^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \quad . \quad . \quad . \quad dt_1 \, dt.$$

Then, since $|K(x, t)| \leqq M$ in R and $|f(t)| \leqq N$ in I, we have

$$|V_n(x)| \leqq |\lambda^n| N M^n \frac{(x-a)^n}{n!} \leqq |\lambda^n| N \frac{[M(b-a)]^n}{n!}, (a \leqq x \leqq b).$$

The series, for which the positive constant $|\lambda^n| N \dfrac{[M(b-a)]^n}{n!}$ is the general expression for the nth term, is convergent for all values of λ, N, M, $(b - a)$. Hence the series (6′) is absolutely and uniformly convergent.

If (5) has a continuous solution, it must be expressed by (6′). If $u(x)$ is continuous in I, its absolute value has a maximum value U. Then

$$|R_{n+1}(x)| \leqq |\lambda^{n+1}| U M^{n+1} \frac{(x-a)^{n+1}}{\underline{|n+1}} \leqq |\lambda^{n+1}| U \frac{[M(b-a)]^{n+1}}{\underline{|n+1}}, \quad (a \leqq x \leqq b).$$

Whence

$$\lim_{n \to \infty} R_{n+1}(x) = 0.$$

Thus we see that the function $u(x)$, satisfying (6), is the continuous function given by the series (6′). As before, we can show that the expression for $u(x)$ given by (6′) satisfies (5). Hence we have the following:

Theorem II.—If

(5) *a*) $u(x) = f(x) + \lambda \int_a^x K(x, t) u(t) dt$ (a, constant).

b) $K(x, t)$ is real and continuous in the rectangle R, for which $a \leqq x \leqq b$, $a \leqq t \leqq b$. $|K(x, t)| \leqq M$ in R, $K(x, t) \not\equiv 0$.

c) $f(x) \not\equiv 0$, is real and continuous in $I: a \leqq x \leqq b$.

d) λ, constant.

then the equation (5) has one and only one continuous solution $u(x)$ in I, and this solution is given by the absolutely and uniformly convergent series (6′).

The results of this article hold without change for the equation

$$u(x) = f(x) + \int_a^x K(x, t)u(t)dt$$

by putting throughout the discussion $\lambda = 1$.

10. Successive Approximations.—We would like to point out that the method of solution by successive approximations differs from that of successive substitutions.

Under the method of successive approximations we select any real function $u_o(x)$ continuous in I. Substitute in the right-hand member of

(1) $$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)dt,$$

in place of $u(t)$, the function $u_o(t)$. We find

$$u_1(x) = f(x) + \lambda \int_a^b K(x, t)u_o(t)dt.$$

The function $u_1(x)$ so determined is real and continuous in I. Continue in like manner by replacing u_o by u_1, and so on. We obtain a series of functions

$$u_o(x),\ u_1(x),\ u_2(x),\ \ldots\ ,\ u_n(x),\ \ldots$$

which satisfy the equations,

(7) $$\begin{cases} u_2(x) = f(x) + \lambda \int_a^b K(x, t)u_1(t)dt \\ \ldots\ldots\ldots\ldots\ldots \\ u_{n-1}(x) = f(x) + \lambda \int_a^b K(x, t)u_{n-2}(t)dt \\ u_n(x) = f(x) + \int_a^b K(x, t)u_{n-1}(t)dt \\ \ldots\ldots\ldots\ldots\ldots \end{cases}$$

From these equations we find

$$(8)\quad u_n(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt$$
$$+ \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) f(t_1) dt_1 dt + \dots$$
$$+ \lambda^{n-1} \int_a^b K(x, t) \int_a^b K(t, t_1) \dots$$
$$\int_a^b K(t_{n-3}, t_{n-2}) f(t_{n-2}) dt_{n-2} \dots dt_1 dt + R_n$$

where

$$R_n = \lambda^n \int_a^b K(x, t) \int_a^b K(t, t_1) \dots \int_a^b K(t_{n-2}, t_{n-1})$$
$$u_o(t_{n-1}) dt_{n-1} \dots dt_1 dt$$

$u_o(x)$ is real and continuous in I and so has a maximum value U in I. Then it is easy to see that

$$|R_n| \leq |\lambda^n| U M^n (b - a)^n.$$

If, then, $|\lambda| M(b - a) < 1$, we have

$$\lim_{n \to \infty} R_n = 0.$$

Thus, as n increases, the series of functions $u_n(x)$ approach a limit function which is given by the series in the right member of (8). We identify this series with the right member of (6′). Thus

$$\lim_{n \to \infty} u_n(x) \equiv u(x).$$

By this process at each step a new function $u_n(x)$ appears dependent upon the choice of $u_o(x)$. We notice, however, that the limit $u(x)$ is independent of the choice of $u_o(x)$.

We can now make an independent proof of the uniqueness of the solution. Suppose there was another solution $v(x)$. Choose $u_o(x) \equiv v(x)$. It is then clear that each $u_n(x)$ will be identical with $v(x)$ and hence the limit will be

$v(x)$. But we have just seen that the limit is independent of the choice of $u_o(x)$. Therefore,

$$v(x) \equiv u(x).$$

A similar discussion can be carried through without further difficulty for the Volterra equation.

11. Iterated Functions.—Place

$$(9) \qquad \begin{cases} K_1(x, t) = K(x, t) \\ K_i(x, t) = \int_a^b K(x, s)K_{i-1}(s, t)ds. \end{cases}$$

The functions $K_1, K_2, \ldots, K_n, \ldots$ so formed are called *iterated functions.*

By successive applications of (9) it is evident that

$$(10) \quad K_i(x, t) = \int_a^b \cdots \int_a^b K(x, s_1)K(s_1, s_2) \cdots K(s_{i-1}, t)ds_{i-1} \cdots ds_1.$$

From (10) $K_n(x, s)$ is an $(n - 1)$-fold integral and $K_p(s, t)$ is a $(p - 1)$-fold integral. Whence we see that $\int_a^b K_n(x, s)K_p(s, t)ds$ is an $(n + p - 1)$-fold integral, which, by some simple changes in the order of integration, is seen to be identical with $K_{n+p}(x, t)$. Hence

$$(11) \qquad K_{n+p}(x, t) = \int_a^b K_n(x, s)K_p(s, t)ds.$$

12. Reciprocal Functions.—Let

$$(12) \quad -k(x, t) = K_1(x, t) + K_2(x, t) + \cdots + K_n(x, t) + \cdots$$

It is easy to show that, when $K(x, t)$ is real and continuous in R, the infinite series for $k(x, t)$ is absolutely and uniformly convergent if $M(b - a) < 1$. Consequently, $k(x, t)$ is real and continuous in R. On account of the first of equations (9) and equation (11), we have

$$-k(x,t) - K(x,t) = K_2(x,t) + K_3(x,t) + \dots + K_n(x,t) + \dots$$
$$= \int_a^b K_1(x,s)K_1(s,t)ds + \dots + \int_a^b K_1(x,s)K_{n-1}(s,t)ds + \dots$$
$$= \int_a^b K_1(x,s)K_1(s,t)ds + \dots + \int_a^b K_{n-1}(x,s)\,K_1(s,t)ds + \dots$$

These equations may be written

$$-k(x,t) - K(x,t) = \int_a^b K_1(x,s)\left[K_1(s,t) + \dots + K_{n-1}(s,t) + \dots\right]ds$$
$$= \int_a^b \left[K_1(x,s) + \dots + K_{n-1}(x,s) + \dots\right]K_1(s,t)ds.$$

If we now make use of (12), we obtain the following characteristic formula:

$$K(x,t) + k(x,t) = \int_a^b K(x,s)k(s,t)ds = \int k(x,s)K(s,t)ds. \tag{13}$$

Two functions $K(x,t)$ and $k(x,t)$ are said to be *reciprocal* if they are both real and continuous in R and if they satisfy the condition (13). A function $k(x,t)$ reciprocal to $K(x,t)$ will exist, provided the series in (12) converges uniformly. But we have seen that this series converges uniformly when $M(b-a) < 1$, where M is the maximum of $|K(x,t)|$ in R. Thus, we have the

Theorem III.—*If $K(x, t)$ is real and continuous in R, there exists a reciprocal function $k(x, t)$ given by (12) provided that*

$$M(b - a) < 1$$

where M is the maximum of $|K(x, t)|$ in R.

13. Volterra's Solution of Fredholm's Equation.—Volterra has shown how to find a solution of

(4) $$u(x) = f(x) + \int_a^b K(x, t)u(t)dt$$

whenever the reciprocal function $k(x, t)$ of $K(x, t)$ is known. If (4) has a continuous solution $u(x)$, then

$$u(t) = f(t) + \int_a^b K(t, t_1)u(t_1)dt_1.$$

Multiplying by $k(x, t)$ and integrating, we find

$$\int_a^b k(x, t)u(t)dt = \int_a^b k(x, t)f(t)dt + \int_a^b \int_a^b k(x, t)K(t, t_1)u(t_1)dt_1dt$$
$$= \int_a^b k(x, t)f(t)dt + \int_a^b \Big[K(x, t_1) + k(x, t_1)\Big]u(t_1)dt_1,$$

which reduces to

(14) $$0 = \int_a^b k(x, t)f(t)dt + \int_a^b K(x, t_1)u(t_1)dt_1.$$

But from (4) we have

$$\int_a^b K(x, t_1)u(t_1)dt_1 = u(x) - f(x).$$

Therefore, (14) may be written

(15) $$u(x) = f(x) - \int_a^b k(x, t)f(t)dt.$$

If (4) has a continuous solution, it is given by this formula and it is unique.

To see that the expression for $u(x)$ given by (15) is, indeed, a solution, we write (15) in the form

$$f(x) = u(x) + \int_a^b k(x, t)f(t)dt.$$

This is an integral equation for the determination of $f(x)$. The function reciprocal to $k(x, t)$ is $K(x, t)$. By what we have just proved, if this equation has a continuous solution, it is unique and is given by

$$f(x) = u(x) - \int_a^b K(x, t)u(t)dt.$$

But this is the equation (1) from which we started. Thus we see that (4) is satisfied by the value of $u(x)$ given by (15). Thus we have the following

Theorem IV.—*If*

a) $K(x, t)$ *is real and continuous in R.* $K(x, t) \not\equiv o$.

b) $f(x)$ *is real and continuous in I.* $f(x) \not\equiv o$.

c) A function $k(x, t)$ *reciprocal to* $K(x, t)$ *exists, then the equation* (4) *has one and only one continuous solution in I and this solution is given by* (15).

The same reasoning applied to (13), considered as an integral equation for the determination of $k(x, t)$, shows that, if a continuous reciprocal function exists, it is unique.

14. Discontinuous Solutions.—We have shown the existence, under proper assumptions, of a unique continuous solution for a linear integral equation of the second kind. This integral equation may have also, in addition, discontinuous solutions. To show this we exhibit the special equation[1]

$$u(x) = \int_o^x t^{x-t}u(t)dt,$$

which has one and only one continuous solution, namely $u(x) \equiv 0$. We can show, by direct substitution, that this

[1] Bôcher, "Integral Equations," p. 17, Cambridge Press, 1909.

equation has also an infinite number of discontinuous solutions given by

$$u(x) = Cx^{x-1},$$

where C is an arbitrary constant not zero.

EXERCISES

Solve the following linear integral equations:

1. a) $u(x) = x + \int_0^x (t - x)u(t)dt.$ *Ans.* $u(x) = \sin x.$

   b) $u(x) = 1 + \int_0^x (t - x)u(t)dt.$ *Ans.* $u(x) = \cos x.$

2. $u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt.u(t)dt.$ *Ans.* $u(x) = x.$

3. $u(x) = \frac{5x}{6} - \frac{1}{9} + \frac{1}{3}\int_0^1 (t + x)u(t)dt.$ *Ans.* $u(x) = x.$

4. $u(x) = 1 + \int_0^x u(t)dt.$ *Ans.* $u(x) = e^x.$

5. $u(x) = e^x - \frac{e}{2} + \frac{1}{2} + \frac{1}{2}\int_0^1 u(t)dt.$ *Ans.* $u(x) = e^x.$

6. $u(x) = \sin x - \frac{x}{4} + \frac{1}{4}\int_0^{\frac{\pi}{2}} tx\,u(t)dt.$ *Ans.* $u(x) = \sin x.$

7. $u(x) = x + \int_0^{1/2} u(t)dt.$ *Ans.* $u(x) = x +$ constant.

8. $u(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x - t) - 4(x - t)^2]u(t)dt.$
   Ans. $u(x) = e^x.$

9. $u(x) = \frac{3}{2}e^x - \frac{xe^x}{2} - \frac{1}{2} + \frac{1}{2}\int_0^1 t\,u(t)dt.$ *Ans.* $u(x) = e^x.$

10. $u(x) = 29 + 6x + \int_0^x (6x - 6t + 5)u(t)dt.$
    Ans. $u(x) = e^{2x} - e^{3x}.$

11. $u(x) = \cos x - x - 2 + \int_0^x (t - x)u(t)dt.$
    Ans. $u(x) = \sin x + x \sin x.$

12. Using the method of successive approximations find five successive approximations in the solution of Exercises 1, 2, 3, 4, 5 after choosing $u_o(x) \equiv 0$.

13. Show that

$$u(x) = A + Bx + \int_o^x [C + D(x - t)]u(t)dt$$

where A, B, C, D are arbitrary constants, has for solution

$$u(x) = K_1 e^{m_1 x} + K_2 e^{m_2 x},$$

where K_1, K_2, m_1, m_2 depend upon A, B, C, D.

14. Show that

$$u(x) = f(x) + \lambda \int_a^b u(t) \left[\sum_1^p \alpha_q(t)\beta_q(t) \right] dt$$

has the solution

$$u(x) = f(x) + \lambda \sum_1^p A_q \alpha_q(x),$$

where the A_q are constants determined by the equation

$$\sum_1^p {}_r A_r \left[\lambda \int_a^b \alpha_r(t)\beta_q(t)dt \right] - A_q = \int_a^b \beta_q(t) f(t) dt \quad (q = 1, \ldots, p).$$

[Heywood-Frèchet].

CHAPTER III

SOLUTION OF FREDHOLM'S EQUATION EXPRESSED AS RATIO OF TWO INTEGRAL SERIES IN λ

15. Fredholm's Equation As Limit of a Finite System of Linear Equations.—The solution given in the previous chapter for the equation

$$(1) \qquad u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)dt$$

has the disadvantage of holding only for restricted values of λ. It is desirable to have, if possible, a solution which holds for all values of λ. Such a solution was given by Fredholm in the form

$$u(x) = \frac{\beta_0(x) + \beta_1(x)\lambda + \cdots}{\alpha_0 + \alpha_1\lambda + \cdots},$$

the numerator and denominator being permanently converging power series in λ.

a) The System of Linear Equations Replacing the Integral Equation.—Before stating explicitly and proving Fredholm's result, we give an outline of the reasoning which led him to his discovery.

Divide the interval (ab) into n equal parts and call the points of division $t_1, t_2, \ldots, t_{n-1}$. Then

$$(2) \quad t_0 = a,\ t_1 = a + h,\ t_2 = a + 2h,\ \ldots,\ t_n = a + nh, \qquad h = \frac{b-a}{n}.$$

Replace the definite integral in (1) by the sum, corresponding to the points of division (2), of which it is the limit. We obtain the approximate equation

$$u(x) - \lambda h[K(x, t_1)u(t_1) + K(x, t_2)u(t_2) + \cdots + K(x, t_n)u(t_n)] = f(x).$$

Since this equation holds for every value of x, it must be satisfied for $x = t_1, t_2, \ldots, t_n$. We thus obtain the following system of n linear equations for the determination of the n unknowns $u(t_1), u(t_2), \ldots, u(t_n)$:

$$(3)\quad \begin{aligned} &u(t_1) - \lambda h[K(t_1, t_1)u(t_1) + K(t_1, t_2)u(t_2) + \cdots + K(t_1, t_n)u(t_n)] = f(t_1) \\ &u(t_2) - \lambda h[K(t_2, t_1)u(t_1) + K(t_2, t_2)u(t_2) + \cdots + K(t_2, t_n)u(t_n)] = f(t_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &u(t_n) - \lambda h[K(t_n, t_1)u(t_1) + K(t_n, t_2)u(t_2) + \cdots + K(t_n, t_n)u(t_n)] = f(t_n). \end{aligned}$$

This system being solved with respect to $u(t_1), u(t_2), \ldots, u(t_n)$, we can plot $u(t_i)$ $(i = 1, \ldots, n)$ as ordinates and by interpolation draw a curve $u(x)$, which we may expect

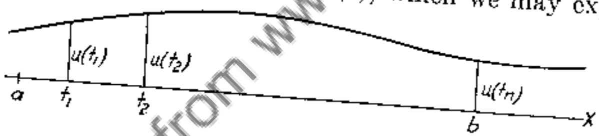


FIG. 2.

to be an approximation to the actual solution. We solve (3), making use of the abbreviations

$$f(t_i) = f_i \quad , \quad u(t_i) = u_i \quad , \quad K(t_i, t_j) = K_{ij}.$$

Denote by Δ the determinant of the coefficients of the u_i in (3). We have

$$\Delta = \begin{vmatrix} 1 - \lambda h K_{11} & -\lambda h K_{12} & \cdots & -\lambda h K_{1n} \\ -\lambda h K_{21} & 1 - \lambda h K_{22} & \cdots & -\lambda h K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda h K_{n1} & -\lambda h K_{n2} & \cdots & 1 - \lambda h K_{nn} \end{vmatrix}.$$

Denote by $\Delta_{\nu\mu}$ the first minor of the element in the νth row and μth column of Δ. Then, from Cramer's formula, we obtain, by solving (3) with respect to u_k,

$$(4) \quad u_k = \frac{\sum_1^n i\, f_i \Delta_{ik}}{\Delta}, \text{ provided } \Delta \neq 0, K = 1, \ldots, n.$$

b) Limit of Δ.—Expanding the determinant Δ, we obtain

$$\Delta = 1 - \lambda \sum_{i=1}^{n} K_{ii} h + \frac{\lambda^2}{2!} \sum_{i,j=1}^{n} h^2 \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} + \cdots$$
$$+ (-1)^n h^n \lambda^n \begin{vmatrix} K_{11} & \ldots & K_{1n} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ K_{n1} & \ldots & K_{nn} \end{vmatrix}.$$

If we now let n increase indefinitely, we see that each term of this series has a definite limit. So that, at least formally,

$$\lim_{n \to \infty} \Delta = 1 - \lambda \int_a^b K(t, t) dt$$
$$+ \frac{\lambda^2}{2!} \int_a^b \int_a^b \begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) \\ K(t_2, t_1) & K(t_2, t_2) \end{vmatrix} dt_1 dt_2$$
$$- \frac{\lambda^3}{3!} \int_a^b \int_a^b \int_a^b \begin{vmatrix} K(t_1, t_1) & \ldots & K(t_1, t_3) \\ \ldots & \ldots & \ldots \\ K(t_3, t_1) & \ldots & K(t_3, t_3) \end{vmatrix} dt_1 dt_2 dt_3 + \cdots$$
$$(5) \quad \equiv D(\lambda).$$

$D(\lambda)$ is called *Fredholm's determinant*, or the determinant of K.

c) Limit of Δ_{ik}.—The expression for $\Delta_{\mu\mu}$ is similar to that for Δ.

$$\Delta_{\mu\mu} = 1 - \lambda \sum_{i=1}^{n}{}' K_{ii} h + \frac{\lambda^2}{2!} \sum_{i,j=1}^{n}{}' h^2 \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} + \cdots,$$

where ' means omit $i = \mu$. Hence

(6) $$\lim_{n \to \infty} \Delta_{\mu\mu} = D(\lambda).$$

Again, from rules for expansion of determinants,

$$\Delta_{\nu\mu} = \lambda h \left\{ K_{\mu\nu} - \lambda \sum_{i=1}^{n} h \begin{vmatrix} K_{\mu\nu} & K_{\mu i} \\ K_{i\nu} & K_{ii} \end{vmatrix} + \frac{\lambda^2}{2!} \sum_{i,j=1}^{n} h_2 \begin{vmatrix} K_{\mu\nu} & K_{\mu i} & K_{\mu j} \\ K_{i\nu} & K_{ii} & K_{ij} \\ K_{j\nu} & K_{ji} & K_{jj} \end{vmatrix} + \cdots \right.$$

Put $h\mathfrak{D}_{\mu\nu} = \Delta_{\nu\mu}$.

If, as n increases indefinitely, we let $(t_\mu t_\nu)$ vary in such a way that $\lim (t_\mu, t_\nu) = (x, y)$, we find, at least formally

(7) $$\lim_{n \to \infty} \mathfrak{D}_{\mu\nu} = \lambda K(x, y) - \lambda^2 \int_a^b \begin{vmatrix} K(x, y) & K(x, t) \\ K(t, y) & K(t, t) \end{vmatrix} dt$$
$$+ \frac{\lambda^3}{2!} \int_a^b \int_a^b \begin{vmatrix} K(x, y) & K(x, t_1) & K(x, t_2) \\ K(t_1, y) & K(t_1, t_1) & K(t_1, t_2) \\ K(t_2, y) & K(t_2, t_1) & K(t_2, t_2) \end{vmatrix} dt_1 dt_2 + \cdots$$

(8) $$\equiv D(x, y; \lambda).$$

This expression for $D(x, y; \lambda)$ is called *Fredholm's first minor.*

d) Limit of u_k.—We can now write (4) in the form

$$u_k = f_k \frac{\Delta_{kk}}{\Delta} + \sum_{i=1}^{n}{}' \frac{f_i \Delta_{ik}}{\Delta}$$
$$= f_k \frac{\Delta_{kk}}{\Delta} + \sum_{i=1}^{n}{}' \frac{f_i h D_{ki}}{\Delta},$$

which in the passage to the limit as $n \to \infty$, becomes

$$u(t_k) = f(t_k) + \frac{1}{D(\lambda)} \int_a^b f(t) D(t_k, t; \lambda) dt.$$

But t_k is any point of division. Then we can replace t_k by x and write

(9) $$u(x) = f(x) + \frac{1}{D(\lambda)} \int_a^b f(t) D(x, t; \lambda) dt.$$

This result has not been obtained by a rigorous mathematical procedure. However, we are inclined to believe that the above expression for $u(x)$ is a solution of (1). This belief is later shown to be correct.

e) Fredholm's Two Fundamental Relations.—We will now develop two relations which will be of use to us later in obtaining a solution of the integral equation (1). We recall the theorem, fundamental in the theory of determinants, that the sum of the products of the elements of any *column* by the corresponding minors of any other column is zero. This theorem applied to the determinant Δ gives

$$(1 - \lambda h K_{jj})\Delta_{jk} - \lambda h K_{kj}\Delta_{kk} - \sum_{i=1}^{n}{}'' \lambda h K_{ij}\Delta_{ik} = 0,$$

where $''$ means omit $i = j, k$. Making use of the relation $\Delta_{\nu\mu} = h\mathfrak{D}(t_\mu, t_\nu)$, we find

$$(1 - \lambda h K_{jj})h\mathfrak{D}_{kj} - \lambda h K_{kj}\Delta_{kk} - \sum_{i=1}^{n}{}'' \lambda h^2 K_{ij}\mathfrak{D}_{ki} = 0.$$

Divide through by h, since $h \neq 0$. The passage to the limit, as $n \to \infty$, gives from the last equation by (6) and (7)

$$D(t_k, t_j; \lambda) - \lambda K(t_k, t_j)D(\lambda) - \lambda \int_a^b K(t, t_j)D(t_k, t; \lambda)dt = 0.$$

This last equation holds for any two points t_j, t_k on the interval (ab). Let us put then $t_k = x$, $t_j = y$ and write

$$(10) \quad D(x, y; \lambda) - \lambda K(x, y)D(\lambda) = \lambda \int_a^b K(t, y)D(x, t; \lambda)dt.$$

This is called Fredholm's *first fundamental relation.*

Now apply the theorem: The sum of the products of the elements of any *row* by the corresponding minors of any other row is zero. This theorem applied to the determinant Δ gives

$$(1 - \lambda h K_{jj})\Delta_{kj} - \lambda h K_{jk}\Delta_{kk} - \sum_{i=1}^{n}{}'' \lambda h K_{ji}\Delta_{ki} = 0.$$

Proceeding as before, we find

$$D(t_j, t_k; \lambda) - \lambda K(t_j, t_k)D(\lambda) - \lambda \int_a^b K(t_j, t)D(t, t_k; \lambda)dt = 0.$$

This equation holds for any two points t_j, t_k on (ab). Let us put then $t_j = x$, $t_k = y$ and write

$$(11)\quad D(x,y;\lambda) - \lambda K(x,y)D(\lambda) - \lambda \int_a^b K(x,t)D(t,y;\lambda)dt = 0.$$

This is Fredholm's *second fundamental relation.*

16. Hadamard's Theorem.—We now proceed to establish rigorously the results of the preceding article. To this end we need a theorem due to Hadamard. To establish this theorem we make use of the following

Lemma.—If all of the elements a_{ik} of the determinant

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix}$$

are real and satisfy the conditions

$$(12)\quad a^2_{r1} + a^2_{r2} + \cdot \cdot \cdot + a^2_{rn} = 1 \quad (r = 1, \ . \ . \ . \ , n),$$

then

$$|A| \leqq 1.$$

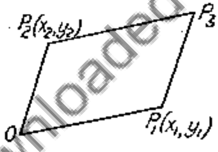


FIG. 3.

We give first two special cases of the lemma which have a geometric interpretation.

1) $n = 2$. The parallelogram $OP_1P_3P_2$ has the vertex O at the origin of a system of rectangular coordinates. The coordinates of P_1 and P_2 are as indicated in the figure. The area A of $OP_1P_3P_2$ is given by

$$A = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

If $OP_1 = OP_2 = 1$, that is, if

$$x_1^2 + y_1^2 = 1 \text{ and } x_2^2 + y_2^2 = 1,$$

then it is geometrically evident that the greatest area is obtained when the figure is a rectangle, and then the area is 1. Hence, we have generally $|A| \leq 1$.

2) $n = 3$. The parallelopiped $OP_1P_2P_3$ has one vertex at the origin of a system of rectangular coordinates. The coordinates of P_1, P_2, P_3 are as indicated in the figure. The volume V of $OP_1P_2P_3$ is given by

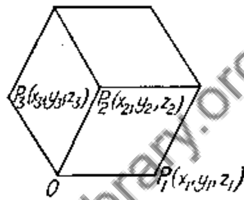


FIG. 4.

$$V = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

If $OP_1 = OP_2 = OP_3 = 1$, that is, if
$x_1^2 + y_1^2 + z_1^2 = 1, x_2^2 + y_2^2 + z_2^2 = 1, x_3^2 + y_3^2 + z_3^2 = 1,$
then it is evident geometrically that the volume is greatest when the figure is a rectangular parallelopiped, in which case the volume is 1. Hence, we have generally

$$|V| \leq 1.$$

Proof of Lemma.—$A(a_{11}, \ldots, a_{nn})$ is a continuous function of its arguments a_{rs} in the region $\mathfrak{A}$ defined by the equations (12). These conditions insure that $|a_{rs}| \leq 1$, and that the region $\mathfrak{A}$ is bounded and closed. Hence A reaches a maximum and a minimum on the region $\mathfrak{A}$. The maximum and minimum which we are seeking are the so-called absolute maximum and minimum. But if a system of values furnishes the absolute maximum (minimum) for A, it furnishes also a relative maximum (minimum). Hence the ordinary methods of the differential calculus can be used for their determination.

Now if a function

$$f(x_1, x_2, \ldots, x_n)$$

of n variables, connected by h distinct relations

$$\phi_1(x_1, \ldots, x_n) = 0, \ \phi_2(x_1, \ldots, x_n) = 0, \ \ldots, \ \phi_h(x_1, \ldots, x_n) = 0$$

has a maximum (minimum), then the n first partial derivatives of the auxiliary function

$$F = f + \lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_h\phi_h,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_h$ are constants, must vanish.[1]

For our problem, $f = A$, $x_r = a_{rs}$

$$\phi_r = \sum_{s=1}^{n} a_{rs}^2 - 1 \quad (r = 1, \ldots, n)$$

and the auxiliary function F becomes

$$F = A + \sum_{s=1}^{n} \frac{\lambda_s}{2}(a_{s1}^2 + a_{s2}^2 + \cdots + a_{sn}^2 - 1).$$

For a maximum (minimum) by the theorem just stated, we must have

$$\frac{\partial F}{\partial a_{jk}} = \frac{\partial A}{\partial a_{jk}} + \lambda_j a_{jk} = 0, \text{ or}$$

(13) $$A_{jk} + \lambda_j a_{jk} = 0, \quad (j, k = 1, 2, \ldots, n),$$[2]

where A_{jk} denotes the minor of a_{jk} in A. Multiply both sides of this equation by a_{jk} and sum with respect to k for $k = 1, 2, \ldots, n$. We obtain

$$A + \lambda_j = 0, \quad \text{since} \sum_{k=1}^{n} a_{jk}^2 = 1$$

or

$$\lambda_j = -A \quad (j = 1, \ldots, n).$$

Substitute this value of λ_j in equation (13). Then

$$A_{jk} = A a_{jk} \qquad (jk = 1, \ldots, n).$$

Whence the determinant

$$\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$$

[1] GOURSAT-HEDRICK, "Mathematical Analysis," vol. 1, §61, Ginn & Co.

[2] For differentiation of determinants, see BALTZER, "Die Determinanten," §3, 14.

adjoint to A, is equal to

$$\begin{vmatrix} Aa_{11} & Aa_{12} & \dots & Aa_{1n} \\ Aa_{21} & Aa_{22} & \dots & Aa_{2n} \\ \dots & \dots & \dots & \dots \\ Aa_{n1} & Aa_{n2} & \dots & Aa_{nn} \end{vmatrix}.$$

The first of these determinants is equal[1] to A^{n-1}. The second reduces to A^{n+1}. Hence

$$A^{n+1} = A^{n-1}$$

But the maximum and minimum of A must satisfy this equation. Therefore, the maximum of A is $+1$, the minimum is -1, and

$$|A| \leqq 1.$$

Hadamard's Theorem.—We are now in a position to prove a more general theorem due to Hadamard: *If the elements b_{ik} of the determinant*

$$B = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}$$

are real and satisfy the inequality

$$|b_{ik}| \leqq M,$$

then

$$|B| \leqq M^n \sqrt{n^n}.$$

Proof.—Let

$$b_{i1}^2 + b_{i2}^2 + \dots + b_{in}^2 = s_i, \ (i = 1, \dots, n)$$

Case I.—Some one or more of the s_i vanish, say $s_k = 0$. Then $b_{ki} = 0$ $(i = 1, \dots, n)$. Therefore $B = 0$, and the theorem is proved in this case.

Case II.—None of the s_i vanish. Then each s_i is positive. That is

$$s_1 > 0, s_2 > 0, \dots, s_n > 0.$$

[1] Baltzer, *Loc. cit.*, §6, 1.

We now consider the determinant

$$\frac{B}{\sqrt{s_1 s_2 \ldots s_n}} = \begin{vmatrix} \frac{b_{11}}{\sqrt{s_1}} & \ldots & \frac{b_{1n}}{\sqrt{s_1}} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \frac{b_{n1}}{\sqrt{s_n}} & \ldots & \frac{b_{nn}}{\sqrt{s_n}} \end{vmatrix}$$

for which $\frac{b_{i1}^2}{s_i} + \ldots + \frac{b_{in}^2}{s_i} = 1. \quad (i = 1, \ldots, n).$

This determinant satisfies all of the conditions of our lemma. Hence

$$|B| \leqq \sqrt{s_1 s_2 \ldots s_n}.$$

But, since $|b_{ik}| \leqq M,$

we have from $s_i = b_{i1}^2 + \ldots + b_{in}^2$

$$s_i \leqq nM^2. \quad (i = 1, \ldots, n)$$

Therefore, $|B| \leqq M^n\sqrt{n^n}.$

17. Convergence Proof.—With the help of Hadamard's theorem we can now prove the convergence of the series for $D(\lambda)$ and $D(x, y; \lambda)$.

a) Convergence of $D(\lambda)$.—$D(\lambda)$ is given by the series

(14) $$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n, \text{ where}$$

(15) $$A_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1\, t_1) \ldots K(t_1\, t_n) \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ K(t_n\, t_1) \ldots K(t_n\, t_n) \end{vmatrix} dt_1 \ldots dt_n \ (n > 0).$$

We have assumed in §8 that $|K(x, t)| \leqq M$ in R. Thus the determinant in the expression for A_n satisfies all of the conditions of Hadamard's theorem, and hence

$$|A_n| \leqq \int_a^b \ldots \int_a^b M^n\sqrt{n^n}\, dt_1 \ldots dt_n$$

$$= \sqrt{n^n} M^n (b - a)^n.$$

Then

$$\left|(-1)^n\frac{\lambda^n}{n!}A_n\right| \leq M^n(b-a)^n|\lambda^n|\frac{\sqrt{n^n}}{n!} \equiv C_n.$$

This identity defines C_n.

We now proceed to show that the series of which C_n is the general term converges. Applying the ratio test, we obtain

$$\frac{C_{n+1}}{C_n} = M(b-a)|\lambda|\sqrt{\left(1+\frac{1}{n}\right)^n}\frac{1}{\sqrt{n+1}}.$$

This ratio has for its limit zero as n becomes infinite. Hence the series of which C_n is the general term converges. Consequently, the series for $D(\lambda)$ is absolutely and permanently convergent. We state this result in the following:

Theorem I.—*The series $D(\lambda)$ is an absolutely and permanently converging power series in λ.*

b) Convergence of $D(x, y; \lambda)$.—We have

(16) $D(x, y; \lambda) = \lambda K(x, y) + \sum_{n=1}^{\infty}(-1)^n\frac{\lambda^{n+1}}{n!}B_n(x, y)$, where

(17) $$B_n(x, y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x, y) & K(x, t_1) & \ldots & K(x, t_n) \\ K(t_1, y) & K(t_1, t_1) & \ldots & K(t_1, t_n) \\ \ldots & \ldots & \ldots & \ldots \\ K(t_n, y) & K(t_n, t_1) & \ldots & K(t_n, t_n) \end{vmatrix} dt_1 \ldots dt_n.$$

It is sometimes convenient to write

$$D(x, y; \lambda) = \sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{n+1}}{n!}B_n(x, y),$$

where we consider $B_o(x, y) = K(x, y)$ and $0! = 1$. The determinant in the expression for B_n satisfies all of the conditions of Hadamard's theorem and hence

$$|B_n| \leq \int_a^b \cdots \int_a^b \sqrt{(n+1)^{n+1}}M^{n+1}dt_1 \ldots dt_n = \sqrt{(n+1)^{n+1}}M^{n+1}(b-a)^n$$

Then

$$\left|(-1)^n \frac{\lambda^{n+1}}{n!} B_n\right| \leqq M^{n+1}(b-a)^n |\lambda|^{n+1} \frac{\sqrt{(n+1)^{n+1}}}{n!} = E_n.$$

This identity defines E_n. We now proceed to show that the series of which E_n is the general term converges. Applying the ratio test,

$$\frac{E_n}{E_{n-1}} = M(b-a)|\lambda| \sqrt{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\sqrt{n+1}} \cdot \frac{n+1}{n}.$$

The ratio has for its limit zero as n becomes infinite. Hence the series of which E_n is the general term converges. Consequently, the series for $D(x, y; \lambda)$ is absolutely and permanently convergent in λ and, moreover, uniformly convergent in x and y on R. Hence the

Theorem II.—*The series $D(x, y; \lambda)$ converges absolutely and permanently in λ, and, moreover, uniformly*[1] *on R: $a \leqq x \leqq b$, $a \leqq y \leqq b$.*

18. Fredholm's Two Fundamental Relations.—We will now proceed to prove the two relations:

$$(18)\quad D(x, y; \lambda) - \lambda K(x, y) D(\lambda) = \lambda \int_a^b D(x, t; \lambda) K(t, y) dt$$

$$(19)\quad = \lambda \int_a^b K(x, t) D(t, y; \lambda)\, dt$$

wihch were previously heuristically obtained.

a) Relation between the Coefficients $A_n(x, y)$ and $B_n(x, y)$.—Substitute in (18), in place of $D(x, y; \lambda)$ and $D(\lambda)$, their series expressions. Then both sides of the equality in (18) become power series in λ. Hence, if (18) is true, the coefficients of corresponding powers of λ on the two sides must be equal. Conversely, if we can show that the coefficients of corresponding powers of λ on the two sides are equal, then (18) will be established. Making the substitutions, we obtain

[1] GOURSAT-HEDRICK, "Mathematical Analysis," §173, note.

$$\lambda K(x, y) + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x, y)$$
$$- \lambda K(x, y)\left[1 + \sum_{n=1}^{\infty} (-1)^n A_n \frac{\lambda^n}{n!}\right]$$
$$= \lambda \int_a^b K(t, y)\left[\lambda K(x, t) + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x, t)\right] dt.$$

On the right-hand side, the series in the integrand is uniformly convergent and remains so when multiplied by $K(t, y)$. Therefore, we can integrate by terms and write

$$\lambda^2 \int_a^b K(x, t)K(t, y)dt + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+2}}{n!} \int_a^b B_n(x, t)K(t, y)dt.$$

If in this second integral we put $n' = n + 1$ and then drop the prime, we obtain

$$\sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x, y) - K(x, y) \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} A_n$$
$$= \lambda^2 \int_a^b K(x, t)K(t, y)dt$$
$$+ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\lambda^{n+1}}{(n-1)!} \int_a^b B_{n-1}(x, t)K(t, y)dt.$$

Compare now the coefficients of corresponding powers of λ on the two sides. We obtain

$$(20) \quad B_n(x, y) = A_n \cdot K(x, y) - n \int_a^b B_{n-1}(x, t)K(t, y)dt.$$

If we establish the truth of this equality, then (18) will be shown to be true.

The relation (19) treated in exactly the same way leads to the relation

$$(21) \quad B_n(x, y) = A_n \cdot K(x, y) - n \int_a^b K(x, t)B_{n-1}(t, y)dt.$$

If we can establish the truth of this equality, then (19) will be shown to be true.

b) Proof of (20) *and* (21).—We prove (20) and also (21) by showing that the explicit expression for $B_n(x, y)$ can by proper expansion and change in notation be written in the form indicated.

The explicit expression for $B_n(x, y)$ is

$$B_n(x, y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x, y) & K(x, t_1) & \cdots & K(x, t_n) \\ K(t_1, y) & K(t_1, t_1) & \cdots & K(t_1, t_n) \\ \cdot & \cdot & \cdots & \cdot \\ K(t_n, y) & K(t_n, t_1) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1 \cdots dt_n.$$

Develop the determinant which appears in the integrand in terms of the elements of the first *column*. We obtain

$$B_n(x, y) = \int_a^b \cdots \int_a^b K(x, y) \times \begin{vmatrix} K(t_1, t_1) & \cdots & K(t_1, t_n) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K(t_n, t_1) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1 \cdots dt_n$$

$$+\sum_{i=1}^{n} (-1)^i \int_a^b \cdots \int_a^b K(t_i, y) \times \begin{vmatrix} K(x, t_1) & \cdots & K(x, t_n) \\ K(t_1, t_1) & \cdots & K(t_1, t_n) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K(t_{i-1}, y) & \cdots & K(t_{i-1}, t_n) \\ K(t_{i+1}, y) & \cdots & K(t_{i+1}, t_n) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K(t_n, y) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1 \cdots dt_n.$$

The first term on the right reduces to $K(x, y)\cdot A_n$, according to (15). In the terms which occur in the summation in place of

$$t_i,\ t_{i+1},\ t_{i+2},\ \dots,\ t_n$$

write

$$t,\ t_i,\ t_{i+1},\ \dots,\ t_{n-1},$$

which means simply a change in the notation of the variables of integration in a multiple definite integral. We obtain

$$\sum_{i=1}^{n}(-1)^i \int_a^b \dots \int_a^b K(t, y) \times \begin{vmatrix} K(x, t_1) & \dots & K(x, t_{i-1}) K(x, t) K(x, t_i) & \dots & K(x, t_{n-1}) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ K(x, t_{n-1}) & \dots & \dots & \dots & K(t_{n-1}, t_{n-1}) \end{vmatrix} dt\, dt_1 \dots dt_{n-1},$$

which may be written by bringing t into the first column

$$\sum_{i=1}^{n}(-1)^{2i-1} \int_a^b \dots \int_a^b K(t, y) \times \begin{vmatrix} K(x, t) & K(x, t_1) & \dots & K(x, t_{n-1}) \\ K(t_1, t) & K(t_1, t_1) & \dots & K(t_1, t_{n-1}) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(t_{n-1}, t) & K(t_{n-1}, t_1) & \dots & K(t_{n-1}, t_{n-1}) \end{vmatrix} dt\, dt_1 \dots dt_{n-1}.$$

This last expression shows that all of the n terms of the summation are equal. Furthermore, the integrations may be performed in any order. We then integrate first with respect to $t_1, \dots, t_{n-1}$. For these integrations, $K(t, y)$ may be regarded as a constant factor and may be taken before the $(n - 1)$ — fold integral, so that we obtain

$$- n \int_a^b K(t, y) \left\{ \int_a^b \cdots \int_a^b \begin{vmatrix} K(x, t) & K(x, t_1) & \cdots & K(x, t_{n-1}) \\ K(t_1, t) & K(t_1, t_1) & \cdots & K(t_1, t_{n-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_{n-1}, t) & K(t_{n-1}, t_1) & \cdots & K(t_{n-1}, t_{n-1}) \end{vmatrix} dt_1 \ldots dt_{n-1} \right\} dt,$$

which, according to (17), may be written

$$- n \int_a^b B_{n-1}(x, t) K(t, y) dt.$$

Hence the development of $B_n(x, y)$ gives

$$B_n(x, y) = A_n \cdot K(x, y) - n \int_a^b B_{n-1}(x, t) \, K(t, y) dt,$$

which proves relation (20). Therefore (18), which is Fredholm's first relation, is true.

Take again the explicit expression for $B_n(x, y)$, but this time develop the determinant in the integrand in terms of the elements of the first *row*. Then, proceeding as before, we obtain for the development of $B_n(x, y)$

$$B_n(x, y) = A_n \cdot K(x, y) - n \int_a^b K(x, t) B_{n-1}(t, y) dt,$$

which proves the relation (21). Therefore, (19), which is Fredholm's second relation, is true. Thus we have established the following.

Theorem III.—*Between Fredholm's determinant $D(\lambda)$ and Fredholm's first minor $D(x, y; \lambda)$ the following double relation holds:*

(18) $D(x, y; \lambda) - \lambda K(x, y) D(\lambda) =$

$$\lambda \int_a^b D(x, t; \lambda) \, K(t, y) dt$$

(19)

$$= \lambda \int_a^b K(x, t) D(t, y; \lambda) dt$$

for all values of λ and for all values of x and y on R.

19. Fredholm's Solution of the Integral Equation When $D(\lambda) \neq 0$.—Fredholm's two fundamental relations, the equations (18) and (19), enable us now to obtain a solution of the integral equation

$$(1) \qquad u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt.$$

We obtain a hint as to the method of procedure from the method of solving the finite system

$$(2) \qquad u_i - \lambda h \sum_{j=1}^{n} K_{ij}u_j = f_i \quad (i = 1, \ldots, n).$$

To find u_k from this system we first multiply by Δ_{ik} and sum with respect to i. We obtain

$$\sum_{i=1}^{n} u_i \Delta_{ik} - \lambda h \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij}u_j\Delta_{ik} = \sum_{i=1}^{n} f_i \Delta_{ik}$$

whence

$$(4) \qquad \Delta u_k = \sum_{i=1}^{n} f_i \Delta_{ik}.$$

Now

$$\Delta_{ik} = h\mathfrak{D}_{ki} \text{ by definition (see §15, } c)$$

and

$$\lim_{h \to 0} \sum_{i=1}^{n} h\mathfrak{D}_{ki} = \int_a^b D(x, t; \lambda)dt \text{ by (7)}.$$

Let us now follow the analogy. Write (1) in the form

$$u(t) = f(t) + \lambda \int_a^b K(t, \xi)u(\xi)d\xi.$$

Multiply both sides of this equation, which we suppose is satisfied by a continuous function u, by $D(x, t; \lambda)$ and then integrate with respect to t from a to b. We obtain

$$(22) \quad \int_a^b D(x, t; \lambda)u(t)dt = \int_a^b D(x, t; \lambda)f(t)dt + \lambda \int_a^b \int_a^b D(x, t; \lambda)K(t, \xi)u(\xi)d\xi dt.$$

The integrand of the double integral being continuous in x and ξ, we can interchange the order of integration[1] in the double integral and write it

$$\int_a^b u(\xi)\left[\lambda\int_a^b D(x, t; \lambda)K(t, \xi)dt\right]d\xi,$$

which, according to (18), becomes

$$\int_a^b\left[D(x, \xi; \lambda) - \lambda D(\lambda)K(x, \xi)\right]u(\xi)d\xi.$$

Hence (22) may be written

$$\int_a^b D(x, t; \lambda)u(t)dt = \int_a^b D(x, t; \lambda)f(t)dt + \int_a^b D(x, \xi; \lambda)\ u(\xi)d\xi - \lambda D(\lambda)\int_a^b K(x, \xi)u(\xi)d\xi,$$

which, on account of (1), reduces to

$$O = \int_a^b D(x, t; \lambda)f(t)dt - D(\lambda)\left[u(x) - f(x)\right].$$

We solve now for $u(x)$, under the assumption $D(\lambda) \neq 0$. We obtain

(23) $$u(x) = f(x) + \int_a^b \frac{D(x, t; \lambda)f(t)}{D(\lambda)}dt.$$

Hence, if u is a continuous function of x which satisfies (1), and if $D(\lambda) \neq 0$, then $u(x)$ is given by (23).

It remains for us to show that also, conversely, the expression for $u(x)$ given by (23) is a solution of (1). We do this by direct substitution. Substitute the value of $u(x)$ as given by (23) in (1). We obtain

$$f(x) + \int_a^b \frac{D(x, t; \lambda)f(t)}{D(\lambda)}dt = f(x) + \lambda\int_a^b K(x, t) \times \left\{f(t) + \int_a^b \frac{D(t, \xi; \lambda)f(\xi)}{D(\lambda)}d\xi\right\}dt.$$

[1] GOURSAT-HEDRICK, "Mathematical Analysis," vol. 1, §123.

Break the last term up into two parts and in the double integral change the order of integration. We obtain

$$\int_a^b \frac{D(x, t; \lambda)f(t)}{D(\lambda)}dt = \lambda \int_a^b K(x,t)f(t)dt + \frac{1}{D(\lambda)}\int_a^b f(\xi)\left[\lambda \int_a^b K(x, t)D(t, \xi; \lambda)dt\right]d\xi,$$

which, according to (19), may be written

$$\int_a^b \frac{D(x, t; \lambda)f(t)}{D(\lambda)}dt = \lambda \int_a^b K(x, t)f(t)dt + \frac{1}{D(\lambda)}\int_a^b f(\xi)\left[D(x, \xi; \lambda) - \lambda K(x, \xi)D(\lambda)\right]d\xi.$$

But this last equation is seen to be an identity. Consequently, the expression for $u(x)$ given by (23) satisfies equation (1). Thus we have proved the following theorem, which is called *Fredholm's first fundamental theorem:*

Theorem IV.—*If*

(a) $D(\lambda) \neq 0$.
(b) $K(x, t)$ *is continuous in* R.
(c) $f(x)$ *is continuous in* I.

then the equation

(1) $$u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$$

has one and only one continuous solution given by

(23) $$u(x) = f(x) + \int_a^b \frac{D(x, t; \lambda)f(t)}{D(\lambda)}dt$$

where $D(x, t; \lambda)$ *and* $D(\lambda)$ *are absolutely and permanently convergent integral series in* λ, *and* $D(x, t; \lambda)$ *converges uniformly with respect to* x *and* t *on* R: $a \leqq x \leqq b$, $a \leqq y \leqq b$.

We have at once, for the special case $f \equiv 0$, the following:

Corollary.—If $D(\lambda) \neq 0$, *then the homogeneous equation*

(24) $$u(x) = \lambda \int_a^b K(x, t)u(t)dt$$

has one and only one continuous solution given by $u(x) \equiv 0$.

Let us point out the analogy with the finite system of linear equations

$$u_i - \lambda h \sum_{j=1}^{n} K_{ij} u_j = f_i \quad (i = 1, \ldots, n)$$

with determinant Δ.

If $\Delta \neq 0$, then this system has one and only one solution. If $f_i \equiv 0$, then the only solution is the trivial one

$$u_1 = \ldots = u_n \equiv 0.$$

But the limit of Δ was $D(\lambda)$. Hence the results of Theorem IV and its corollary are exactly what was to be expected from the analogy with the finite system.

20. Solution of the Homogeneous Equation When $D(\lambda) = 0$, $D'(\lambda) \neq 0$.—The discussion up to this point has been made under the assumption $D(\lambda) \neq 0$. Let us now see what happens when $D(\lambda) = 0$, first with respect to the homogeneous equation (24).

Let λ_o be a value of λ for which

(25) $$D(\lambda_o) = 0.$$

We now consider the solution of the homogeneous integral equation (24) for this particular value of λ:

(26) $$u(x) = \lambda_o \int_a^b K(x, t) u(t) dt.$$

We obtain a solution of (26) by means of Fredholm's second fundamental relation (19), which is true for all values of λ and hence for $\lambda = \lambda_o$. With this value of λ, on account of (25), equation (19) becomes

$$D(x, y; \lambda_o) = \lambda_o \int_a^b K(x, t) D(t, y; \lambda_o) dt.$$

The equality holds for every value of y on the interval (ab) and, therefore, for $y = y_o$. Then

$$D(x, y_o; \lambda_o) = \lambda_o \int_a^b K(x, t) D(t, y_o; \lambda_o) dt,$$

But this is just the equation (26) with $u(x)$ replaced by $D(x, y_o; \lambda_o)$. Thus we see that $u(x) = D(x, y_o; \lambda_o)$ is a solution of (26). Moreover, this solution is continuous,[1] for $D(x, y; \lambda)$ is uniformly convergent in x and y and its terms are continuous. But $D(x, y_o; \lambda_o)$ may be identically zero in x, either on account of an unfortunate choice of y_o, in which case we could choose some other value for y_o, or because $D(x, y; \lambda_o) \equiv 0$ in x and y, in which case the above solution reduces to the trivial one $u \equiv 0$, no matter how we choose y_o. We have thus proved the following:

Theorem V.—*If* $D(\lambda_o) = 0$ *and* $D(x, y; \lambda_o) \not\equiv 0$, *then for a proper choice of* y_o, $u(x) = D(x, y_o; \lambda_o)$ *is a continuous solution of*

$$u(x) = \lambda_o \int_a^b K(x, t)u(t)dt$$

and $u(x) \not\equiv 0$.

In the theorem just stated, the condition $D(x, y; \lambda_o) \not\equiv 0$ may be replaced by the condition $D'(\lambda) \neq 0$. To show this we prove the following

$$\int_a^b D(x, x; \lambda)dx = -\lambda D'(\lambda). \tag{27}$$

We prove (27) by making use of the series expressions for $D'(\lambda)$ and $D(x, y; \lambda)$. We have from (14)

$$D(\lambda) = 1 - \lambda A_1 + \frac{\lambda^2}{2!}A_2 - \frac{\lambda^3}{3!}A_3 + \cdots$$

where the A_n are given by (15). Then

$$D'(\lambda) = -A_1 + \lambda A_2 - \frac{\lambda^2}{2!}A_3 + \cdots$$

$$= -\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{n!}A_{n+1}.$$

[1] Goursat-Hedrick, "Mathematical Analysis," vol. 1, §173.

In the expression for A_{n+1}:

$$A_{n+1} = \int_a^b \cdots \int_a^b \begin{vmatrix} k(t_1, t_1) K(t_1, t_2) & \cdots & K(t_1, t_{n+1}) \\ K(t_2, t_1) K(t_2, t_2) & \cdots & K(t_2, t_{n+1}) \\ \cdots & \cdots & \cdots \\ K(t_{n+1}, t_1) K(t_{n+1}, t_2) & \cdots & K(t_{n+1}, t_{n+1}) \end{vmatrix} dt_1 \cdots dt_{n+1},$$

in place of $\quad t_1, t_2, t_3, \ldots, t_n, \quad t_{n+1}$

put $\quad x, t_1, t_2, \ldots, t_{n-1}, t_n.$

Then

$$A_{n+1} = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x, x)\ K(x, t_1) & \cdots & K(x, t_n) \\ K(t_1, x)\ K(t_1, t_1) & \cdots & K(t_1, t_n) \\ \cdots & \cdots & \cdots \\ K(t_n, x)\ K(t_n, t_1) & \cdots & K(t_n, t_n) \end{vmatrix} dx dt_1 \cdots dt_n.$$

In this multiple definite integral we change the order of integration[1] and integrate first with respect to $dt_1 \ldots dt_n$, whence

$$A_{n+1} = \int_a^b \left\{ \int_a^b \cdots \int_a^b \begin{vmatrix} K(x, x)\ K(x, t_1) & \cdots & K(x, t_n) \\ K(t_1, x)\ K(t_1, t_1) & \cdots & K(t_1, t_n) \\ \cdots & \cdots & \cdots \\ K(t_n, x)\ K(t_n, t_1) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1 \cdots dt_n \right\} dx,$$

which, on account of (17), becomes

$$A_{n+1} = \int_a^b B_n(x, x) dx.$$

[1] GOURSAT-HEDRICK "Mathematical Analysis," vol 1, §123; PIERPONT, JAMES, "The Theory of Functions of Real Variables," vol. I, §570.

Therefore,

$$D'(\lambda) = -\sum_{n=0}^{\infty}(-1)^n \frac{\lambda^n}{n!}\int_a^b B_n(x, x)dx.$$

But

$$\sum_{n=0}^{\infty}(-1)^n \frac{\lambda^n}{n!} B_n(x, x)dx$$

is a series uniformly convergent in x. We can then interchange the order of writing the summation and the integration[1] in the expression for $D'(\lambda)$ and write

$$D'(\lambda) = -\int_a^b \sum_{n=0}^{\infty}(-1)^n \frac{\lambda^n}{n!} B_n(x, x)dx.$$

Multiply both sides by $-\lambda$, and then by (16) we see that, indeed,

$$\int_a^b D(x, x; \lambda)dx = -\lambda D'(\lambda). \tag{27}$$

Let us suppose now that $D(\lambda_o) = 0$ and $D'(\lambda_o) \neq 0$, then certainly $\lambda_o \neq 0$, since $D(0) = 1$. Hence, if we write (27) with $\lambda = \lambda_o$, the right-hand side of this equality is different from zero and so also the left-hand side, and, therefore, $D(x, x; \lambda_o) \not\equiv 0$ in x and, consequently, $D(x, y; \lambda_o) \not\equiv 0$ in x and y. Hence, indeed, in Theorem V the condition $D(x, y; \lambda_o) \not\equiv 0$ may be replaced by $D'(\lambda_o) \neq 0$.

We remark further that if $u(x) = D(x, y_o; \lambda_o)$ is a solution of the homogeneous integral equation (26), then Cu (C, an arbitrary constant) is also a solution. Thus, there are an infinitude of solutions which differ only by a constant factor. We shall later show that there are no other solutions. This is in analogy with the finite system of linear equations

$$u_i - \lambda h \sum_{j=1}^{n} K_{ij}u_j = 0 \qquad (i = 1, \ . \ . \ . \ , n)$$

with determinant Δ. If $\Delta = 0$, while not all of the first minors Δ_{ik} vanish, then these n equations determine uniquely the ratios of the $u_1, \ . \ . \ . \ , u_n$, that is, $u_j = C_j u_n$

[1] GOURSAT-HEDRICK, "Mathematical Analysis", vol. 1, §174.

($j = 1, \ldots, n$; $C_n = 1$). Now, $\Delta = 0$ corresponds to $D(\lambda) = 0$, while *not all* Δ_{ik} *vanish* corresponds to $D(x, y; \lambda_o) \not\equiv 0$.

21. Solution of the Homogeneous Integral Equation When D(λ) = 0.—It remains to consider the case

$$D(\lambda_o) = 0, \; D(x, y; \lambda_o) \equiv 0.$$

which corresponds in the linear system to the case where Δ and all its first minors are zero, and, where it becomes necessary, to consider the minors of higher order of Δ. Accordingly, we have to consider, in the treatment of the integral equation, the limits of these higher minors of Δ as h approaches zero.

In what follows let us use with Heywood-Frechet ("L'Équation De Fredholm," page 53) the notation

$$K\begin{pmatrix} s_1, s_2, & \ldots & , s_n \\ t_1, t_2, & \ldots & , t_n \end{pmatrix} \equiv \begin{vmatrix} K(s_1, t_1) & \ldots & K(s_1, t_n) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ K(s_n, t_1) & \ldots & K(s_n, t_n) \end{vmatrix}.$$

(*a*) *Definition of the pth Minor of D*(λ).—Let

$$(28) \quad B_n\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix} = \int_a^b \ldots \int_a^b K\begin{pmatrix} x_1, & \ldots & , x_p, t_1, & \ldots & , t_n \\ y_1, & \ldots & , y_p, t_1, & \ldots & , t_n \end{pmatrix} dt_1 \ldots dt_n$$

with

$$(29) \quad B_o\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix} = K\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix}.$$

Then the *pth minor of* $D(\lambda)$ is defined by the infinite series

$$(30) \quad D\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix} \lambda = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+p}}{n!} B_n\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix} \equiv D_p(x, y; \lambda),$$

which for $p = 1$ reduces to $D(x, y; \lambda)$.

By means of Hadamard's theorem we prove, exactly as before, the

Theorem VI.—*The infinite series for* $D\begin{pmatrix} x_1, & \ldots, & x_p \\ y_1, & \ldots, & y_p \end{pmatrix} \lambda$ *is absolutely and permanently convergent in* λ, *and uniformly convergent in* $x_1, \ldots, x_p, y_1, \ldots, y_p$ *for* $a \leqq x_\alpha \leqq b$, $a \leqq y_\beta \leqq b$ $(\alpha, \beta = 1, \ldots, p)$.

Corollary.—When two of the x's become equal, say $x_r = x_s$, *or when two of the y's become equal, say* $y_i = y_j$, *then* $D_p(x, y; \lambda)$ *vanishes.* For then in the integrand of $B_n\begin{pmatrix} x_1, & \ldots, & x_p \\ y_1, & \ldots, & y_p \end{pmatrix}$ two rows (columns) become equal. Therefore,

$$B_n\begin{pmatrix} x_1, & \ldots, & x_p \\ y_1, & \ldots, & y_p \end{pmatrix} = 0, \text{ and hence } D_p(x, y; \lambda) = 0.$$

In like manner, if in $D_p(x, y; \lambda)$ two of the x's or two of the y's are interchanged, $D_p(x, y; \lambda)$ changes sign.

b) Generalization of the Two Fundamental Relations.—Expand the determinant in $B_n\begin{pmatrix} x_1, & \ldots, & x_p \\ y_1, & \ldots, & y_p \end{pmatrix}$ according to the elements of the column y_β:

$$B_n\begin{pmatrix} x_1, & \ldots, & x_p \\ y_1, & \ldots, & y_p \end{pmatrix} = \int_a^b \ldots \int_a^b \left\{ \sum_{\alpha=1}^{p} (-1)^{\alpha+\beta} \times \right.$$
$$K(x_\alpha, y_\beta) K\begin{pmatrix} x_1, & \ldots, & x_{\alpha-1}, x_{\alpha+1}, & \ldots, & x_p, t_1, & \ldots, & t_n \\ y_1, & \ldots, & y_{\beta-1}, y_{\beta+1}, & \ldots, & y_p, t_1, & \ldots, & t_n \end{pmatrix}$$
$$+ \sum_{i=1}^{n} (-1)^{p+i+\beta} K(t_i, y_\beta) \times$$
$$\left. K\begin{pmatrix} x_1, & \ldots, & x_p, t_1, & \ldots, & t_{i-1}, t_{i+1}, & \ldots, & t_n \\ y_1, & \ldots, & y_{\beta-1}, y_{\beta+1}, & \ldots, & y_p, t_1, & \ldots, & t_n \end{pmatrix} \right\} dt_1 \ldots dt_n.$$

In the first sum, $K(x_\alpha, y_\beta)$ may be taken before the integral sign, and, according to (28), the sum becomes

$$\sum_{\alpha=1}^{p} (-1)^{\alpha+\beta} K(x_\alpha, y_\beta) B_n\begin{pmatrix} x_1, & \ldots, & x_{\alpha-1}, x_{\alpha+1}, & \ldots, & x_p \\ y_1, & \ldots, & y_{\beta-1}, y_{\beta+1}, & \ldots, & y_p \end{pmatrix}.$$

In the second sum make a change of notation.

In place of $t_i, t_{i+1}, t_{i+2}, \; . \; . \; . \; , t_n$
write $t \;, t_i \;, t_{i+1}, \; . \; . \; . \; , t_{n-1}$

The ith term of the second sum becomes

$$(-1)^{p+i+\beta} K(t, y_\beta) \times$$
$$K\begin{pmatrix} x_1, & . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; , x_p, t_1, & . \; . \; . & . \; . \; . \; . \; . \; . \; . \; . \; , t_{n-1} \\ y_1, & . \; . \; . \; , y_{\beta-1}, y_{\beta+1}, \; . \; . \; . \; , y_p, t_1, & . \; . \; . & , t_{i-1}, t, t_i, \; . \; . \; . \; , t_{n-1} \end{pmatrix}.$$

Bring the column t between the columns $y_{\beta-1}, y_{\beta+1}$ by $i + p - \beta - 1$ transpositions of columns. We get, then, for the ith term of the second sum

$$-K(t, y_\beta) \times$$
$$K\begin{pmatrix} x_1, & . \; . \; . \; x_{\beta-1}, x_\beta, x_{\beta+1}, & . \; . \; . \; , x_p, t_1, & . \; . \; . \; . \; , t_{n-1} \\ y_1, & . \; . \; . \; , y_{\beta-1}, t, y_{\beta+1}, & . \; . \; . \; , y_p, t_1, & . \; . \; . \; . \; , t_{n-1} \end{pmatrix}.$$

Hence, all terms in $\sum_{i=1}^{n}$ are equal and this sum may be written, if, moreover, we integrate first with respect to $t_1, \; . \; . \; . \; , t_{n-1}$,

$$-n \int_a^b K(t, y_\beta) \Bigg\{ \int_a^b \; . \; . \; . \int_a^b$$
$$K\begin{pmatrix} x_1, & . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; , x_p, t_1, & . \; . \; . \; . \; , t_{n-1} \\ y_1, & . \; . \; . \; , y_{\beta-1}, t, y_{\beta+1}, \; . \; . \; . \; , y_p, t_1, & . \; . \; . \; . \; , t_{n-1} \end{pmatrix}$$
$$dt_1 \; . \; . \; . \; dt_{n-1} \Bigg\} dt,$$

which, according to (28), reduces to

$$-n \int_a^b K(t, y_\beta) \times$$
$$B_{n-1}\begin{pmatrix} x_1, & . \; . \; . \; . \; , x_{\beta-1}, x_\beta, x_{\beta+1}, & . \; . \; . \; x_p \\ y_1, & . \; . \; . \; . \; , y_{\beta-1}, t, y_{\beta+1}, & . \; . \; . \; . \; , y_p \end{pmatrix} dt.$$

Thus we arrive at the formula

$$
(31) \qquad B_n\begin{pmatrix} x_1, & \dots & , x_p \\ y_1, & \dots & , y_p \end{pmatrix} =
$$

$$
\sum_{\alpha=1}^{p} (-1)^{\alpha+\beta} K(x_\alpha, y_\beta) B_n\begin{pmatrix} x_1, & \dots & , x_{\alpha-1}, x_{\alpha+1}, & \dots & , x_p \\ y_1, & \dots & , y_{\beta-1}, y_{\beta+1}, & \dots & , y_p \end{pmatrix}
$$

$$
- n \int_a^b K(t, y_\beta) B_{n-1}\begin{pmatrix} x_1, & \dots & , x_{\alpha-1}, x_\alpha, x_{\alpha+1}, & \dots & , x_p \\ y_1, & \dots & , y_{\beta-1}, t, y_{\beta+1}, & \dots & , y_p \end{pmatrix} dt.
$$

In like manner, by expanding the integrand of $B_n\begin{pmatrix} x_1, & \dots & , x_p \\ y_1, & \dots & , y_p \end{pmatrix}$ according to the elements of the *row* x_α, we obtain

$$
(32) \qquad B_n\begin{pmatrix} x_1, & \dots & , x_p \\ y_1, & \dots & , y_p \end{pmatrix} =
$$

$$
\sum_{\beta=1}^{p} (-1)^{\alpha+\beta} K(x_\alpha, y_\beta) B_n\begin{pmatrix} x_1, & \dots & , x_{\alpha-1}, x_{\alpha+1}, & \dots & , x_p \\ y_1, & \dots & , y_{\beta-1}, y_{\beta+1}, & \dots & , y_p \end{pmatrix}
$$

$$
- n \int_a^b K(x_\alpha, t) B_{n-1}\begin{pmatrix} x_1, & \dots & , x_{\alpha-1}, t, x_{\alpha+1}, & \dots & , x_p \\ y_1, & \dots & , y_{\beta-1}, y_\beta, y_{\beta+1}, & \dots & , y_p \end{pmatrix} dt.
$$

If now we multiply both sides of (31) and (32) by $(-1)^n \dfrac{\lambda^{n+p}}{n!}$ and sum with respect to n from $n = 0$ to $n = \infty$, we obtain, on account of the definition given in (30), the following double relation, which is a *generalization of Fredholm's two fundamental relations* (10) and (11).

$$
(33) \qquad D\begin{pmatrix} x_1, & \dots & , x_p \\ y_1, & \dots & , y_p \end{pmatrix} \lambda =
$$

$$
\sum_{\alpha=1}^{p} (-1)^{\alpha+\beta} \lambda K(x_\alpha, y_\beta) D\begin{pmatrix} x_1, & \dots & , x_{\alpha-1}, x_{\alpha+1}, & \dots & , x_p \\ y_1, & \dots & , y_{\beta-1}, y_{\beta+1}, & \dots & , y_p \end{pmatrix} \lambda
$$

$$
+ \lambda \int_a^b K(t, y_\beta) D\begin{pmatrix} x_1, & \dots & , x_{\beta-1}, x_\beta, x_{\beta+1}, & \dots & , x_p \\ y_1, & \dots & , y_{\beta-1}, t, y_{\beta+1}, & \dots & , y_p \end{pmatrix} \lambda \, dt
$$

$$(34) \quad = \sum_{\beta=1}^{p} (-1)^{\alpha+\beta} \lambda K(x_\alpha, y_\beta) D\begin{pmatrix} x_1, & \ldots, & x_{\alpha-1}, x_{\alpha+1}, & \ldots, & x_p \\ y_1, & \ldots, & y_{\beta-1}, y_{\beta+1}, & \ldots, & y_p \end{pmatrix} \lambda \Bigg)$$

$$+ \lambda \int_a^b K(x_\alpha, t) D\begin{pmatrix} x_1, & \ldots, & x_{\alpha-1}, t, x_{\alpha+1}, & \ldots, & x_p \\ y_1, & \ldots, & y_{\alpha-1}, y_\alpha, y_{\alpha+1}, & \ldots, & y_p \end{pmatrix} \lambda \Bigg) dt.$$

c) Relation between $D^{(p)}(\lambda)$ *and* $D\begin{pmatrix} x_1, & \ldots, & x_p \\ y_1, & \ldots, & y_p \end{pmatrix} \lambda \Bigg)$.

The relation (27) between $D'(\lambda)$ and $D(x, x; \lambda)$ generalizes as follows:

$$(35) \quad \int_a^b \cdots \int_a^b D\begin{pmatrix} x_1, & \ldots, & x_p \\ x_1, & \ldots, & x_p \end{pmatrix} \lambda \Bigg) dx_1 \ldots dx_p = (-1)^p \lambda^p \frac{d^p D(\lambda)}{d\lambda^p}.$$

Proof.—From the series expression for $D(\lambda)$:

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n,$$

we get

$$D^{(p)}(\lambda) = \sum_{n=p}^{\infty} (-1)^n \frac{\lambda^{n-p}}{(n-p)!} A_n,$$

which, after the change in notation $n - p = n'$ and a final dropping of the prime ($'$), may be written

$$D^{(p)}(\lambda) = \sum_{n=0}^{\infty} (-1)^{n+p} \frac{\lambda^n}{n!} A_{n+p}.$$

But by (15)

$$A_{n+p} = \int_a^b \cdots \int_a^b K\begin{pmatrix} t_1, & \ldots, & t_{n+p} \\ t_1, & \ldots, & t_{n+p} \end{pmatrix} dt_1 \ldots dt_{n+p}.$$

If now

in place of $\quad t_1, \ldots, t_p, t_{p+1}, \ldots, t_{p+n}$

we write $\quad x_1, \ldots, x_p, t_1, \ldots, t_n$

and then change the order of integration, as we may, so that we integrate first with respect to $t_1, \ . \ . \ . \ , t_n$, we obtain

$$A_{n+p} = \int_a^b \cdots \int_a^b \left\{ \int_a^b \cdots \int_a^b K\begin{pmatrix} x_1, & \ldots, & x_p, & t_1, & \ldots, & t_n \\ x_1, & \ldots, & x_p, & t_1, & \ldots, & t_n \end{pmatrix} dt_1 \ldots dt_n \right\} dx_1 \ldots dx_p,$$

which, on account of (28), becomes

$$A_{n+p} = \int_a^b \cdots \int_a^b B_n\begin{pmatrix} x_1, & \ldots, & x_p \\ x_1, & \ldots, & x_p \end{pmatrix} dx_1 \ldots dx_p.$$

Multiply both sides of this equality by $(-1)^n \dfrac{\lambda^{n+p}}{n!}$ and sum with respect to n from $n = 0$ to $n = \infty$. We obtain

$$(-1)^p \lambda^p \frac{d^p D(\lambda)}{d\lambda^p} = \sum_{n=0}^{\infty} \int_a^b \cdots \int_a^b (-1)^n \frac{\lambda^{n+p}}{n!} B_n\begin{pmatrix} x_1, & \ldots, & x_p \\ x_1, & \ldots, & x_p \end{pmatrix} dx_1 \ldots dx_p.$$

It is permissible here to put the summation under the multiple integral sign. If we then make use of the equation of definition (30), we obtain

$$(-1)^p \lambda^p \frac{d^p D(\lambda)}{d\lambda^p} = \int_a^b \cdots \int_a^b D\begin{pmatrix} x_1, & \ldots, & x_p \\ x_1, & \ldots, & x_p \end{pmatrix} \lambda \Bigg) dx_1 \ldots dx_p,$$

which establishes the equality (35).

We make use of this result to prove that not all of the Fredholm minors vanish. Let λ_o be a root of $D(\lambda) = 0$. Then certainly $\lambda_o \neq 0$ for $D(0) = 1$. Furthermore, λ_o is a root of $D(\lambda)$ of finite *multiplicity* r $(r \geqq 1)$, defined by

$$D(\lambda_o) = 0, D'(\lambda_o) = 0, \ . \ . \ . \ , D^{(r-1)}(\lambda_o) = 0, D^{(r)}(\lambda_o) \neq 0.$$

The multiplicity r of the root must be finite, otherwise

$$D(\lambda) \equiv 0.$$

In (35) put $\lambda = \lambda_o$ and $p = r$, then the right member by our hypothesis does not vanish. Therefore, the left member does not vanish. Hence, since $\lambda_o \neq 0$,

$$D\begin{pmatrix} x_1, & . & . & . & . & , x_r \\ x_1, & . & . & . & . & , x_r \end{pmatrix} \lambda_o \not\equiv 0 \text{ in } x_1, \quad . \quad . \quad . \quad . \quad , x_r$$

and, consequently,

$$D\begin{pmatrix} x_1, & . & . & . & . & , x_r \\ y_1, & . & . & . & . & , y_r \end{pmatrix} \lambda_o \not\equiv 0 \text{ in } x_1, \quad . \quad . \quad . \quad . \quad , x_r, y_1, \quad . \quad . \quad . \quad , y_r.$$

Hence, in the series

$$D(\lambda_o) = 0,\ D\begin{pmatrix} x \\ y \end{pmatrix} \lambda_o,\ D\begin{pmatrix} x_1, x_2 \\ y_1, y_2 \end{pmatrix} \lambda_o,\ D\begin{pmatrix} x_1, x_2, x_3 \\ y_1, y_2, y_3 \end{pmatrix} \lambda_o, \quad . \quad . \quad . \quad .$$

we must come to a number $q \leqq r$, called the *index of* λ_o, such that

$$D(\lambda_o) = 0,\ D(x, y; \lambda_o) \equiv 0, \quad . \quad . \quad . \quad ,$$

$$D\begin{pmatrix} x_1, & . & . & . & . & , x_{q-1} \\ y_1, & . & . & . & . & , y_{q-1} \end{pmatrix} \lambda_o \equiv 0,\ D\begin{pmatrix} x_1, & . & . & . & . & , x_q \\ y_1, & . & . & . & . & , y_q \end{pmatrix} \lambda_o \not\equiv 0.$$

That is, there exists a particular set of values x_1', ... , x_q', y_1', ... , y_q' for the variables x_1, ... , x_q, y_1, ... , y_q such that we have the following numerical inequality:

$$D\begin{pmatrix} x_1', & . & . & . & . & , x_q' \\ y_1', & . & . & . & . & , y_q' \end{pmatrix} \lambda_o \neq 0.$$

Incidentally, we have proved the

Theorem VII.—*The index q of a root λ_o of $D(\lambda)$ is at most equal to the multiplicity r of λ_o: $q \leqq r$.*

d) The q Independent Solutions of the Homogeneous Equation.—Let λ_o be a root of $D(\lambda) = 0$ of index q, so that $D(\lambda_o) = 0$, $D(x, y; \lambda_o) \equiv 0$, ... ,

$$D\begin{pmatrix} x_1, & . & . & . & . & , x_{q-1} \\ y_1, & . & . & . & . & , y_{q-1} \end{pmatrix} \lambda_o \equiv 0$$

but

$$D\begin{pmatrix} x_1, & . & . & . & . & , x_q \\ y_1, & . & . & . & . & , y_q \end{pmatrix} \lambda_o \not\equiv 0.$$

Write down the second of Fredholm's generalized fundamental relations (34) for $\lambda = \lambda_o$, $p = q$

$$x_1 = x_1', \quad . \quad . \quad . \quad . \quad , x_{\alpha-1} = x'_{\alpha-1}, x_\alpha = x, x_{\alpha+1} = x'_{\alpha+1}, \quad . \quad . \quad . \quad , x_q = x_q'$$

$$y_1 = y_1', \quad . \quad . \quad . \quad . \quad , y_{\beta-1} = y'_{\beta-1}, y_\beta = y_\beta', y_{\beta+1} = y'_{\beta+1}, \quad . \quad . \quad . \quad , y_q = y_q'$$

Then

$$D\begin{pmatrix} x_1', & . \; . \; . \; . \; , & x'_{\alpha-1}, & x, & x'_{\alpha+1}, & . \; . \; . \; . \; , & x_q' \\ y_1', & . \; . \; . \; . \; , & y'_{\beta-1}, & y_\beta', & y'_{\beta+1}, & . \; . \; . \; . \; , & y_q' \end{pmatrix} \lambda_o \Bigg) =$$

$$\lambda_o \int_a^b K(x, t) D\begin{pmatrix} x_1', & . \; . \; . \; , & x'_{\alpha-1}, & t, & x'_{\alpha+1}, & . \; . \; . \; , & x_q' \\ y_1', & . \; . \; . \; , & y'_{\beta-1}, & y_\beta', & y'_{\beta+1}, & . \; . \; . \; , & y_q' \end{pmatrix} \lambda_o \Bigg) dt,$$

since by hypothesis

$$D\begin{pmatrix} x_1, & . \; . \; . \; . \; , & x_{\alpha-1}, & x_{\alpha+1}, & . \; . \; . \; . \; , & x_q \\ y_1, & . \; . \; . \; . \; , & y_{\beta-1}, & y_{\beta+1}, & . \; . \; . \; . \; , & y_q \end{pmatrix} \lambda_o \Bigg) \equiv 0.$$

Hence, if we divide by $D\begin{pmatrix} x_1', & . \; . \; . \; . \; , & x_q' \\ y_1', & . \; . \; . \; . \; , & y_q' \end{pmatrix} \lambda_o \Bigg)$

and put

$$(36) \quad D\begin{pmatrix} x_1', & . \; . \; . \; . \; , & x'_{\alpha-1}, & x \;, & x'_{\alpha+1}, & . \; . \; . \; . \; , & x_q' \\ y_1', & . \; . \; . \; . \; , & y'_{\beta-1}, & y'_\beta, & y'_{\beta+1}, & . \; . \; . \; . \; , & y_q' \end{pmatrix} \lambda_o \Bigg) = \varphi_\alpha(x, \lambda_o) D\begin{pmatrix} x_1', & . \; . \; . \; . \; , & x_q' \\ y_1', & . \; . \; . \; . \; , & y_q' \end{pmatrix} \lambda_o \Bigg)$$

we have

$$\varphi_\alpha(x, \lambda_o) = \lambda_o \int_a^b K(x, t) \varphi_\alpha(t, \lambda_o) dt,$$

which expresses that the q functions

$$\varphi_1(x, \lambda_o), \varphi_2(x, \lambda_o), \quad . \quad . \quad . \quad , \varphi_q(x, \lambda_o)$$

are solutions of the homogeneous equation (26). These solutions are continuous and

$$(37) \qquad \varphi_\alpha(x_\beta', \lambda_o) = \begin{cases} 1, & \beta = \alpha \\ 0, & \beta \neq \alpha \end{cases} .$$

from (30) and (28). Furthermore, these functions are linearly independent. That is, if a relation of the form

$$C_1\varphi_1(x) + C_2\varphi_2(x) + \cdots + C_q\varphi_q(x) \equiv 0$$

exists, where $C_1, \ldots, C_q$ are constants, then we must have $C_1 = C_2 = \cdots = C_q = 0$. For, from (37), if $x = x_\alpha'$, then $C_\alpha = 0$.

From the homogeneity of (26) it follows that

$$u(x) = C_1\varphi_1(x) + \cdots + C_q\varphi_q(x) \tag{38}$$

is again a solution of (26) for arbitrary values of $C_1, \ldots, C_q$. We thus have a q-fold infinitude of solutions.

e) Completeness Proof.—It remains to show that every solution of (26) can be put in the form (38).

If $v(x)$ is any solution of (26), then

$$v(x) = \lambda_o \int_a^b K(x, t)v(t)dt.$$

Whence

$$0 \equiv \int_a^b H(x, t)\left\{v(t) - \lambda_o \int_a^b K(t, s)v(s)ds\right\}dt,$$

where $H(x, t)$ is any continuous function. On subtracting the second equation from the first, we obtain

$$v(x) = \lambda_o \int_a^b N(x, t)v(t)dt \tag{39}$$

where

$$\lambda_o N(x, t) = \lambda_o K(x, t) - \left\{H(x, t) - \lambda_o \int_a^b H(x, s)K(s, t)ds\right\}.$$

Now apply (33) with $p = q + 1$, $x_{q+1} = x$, $y_{q+1} = y$ and notice that a transposition of two x's or two y's changes the sign of D. Then

$$D\begin{pmatrix} x, x_1, \ldots, x_q \\ y, y_1, \ldots, y_q \end{pmatrix}\lambda_o = \lambda K(x, y) D\begin{pmatrix} x_1, \ldots, x_q \\ y_1, \ldots, y_q \end{pmatrix}\lambda$$
$$-\sum_{\alpha=1}^{q} \lambda K(x_\alpha, y) D\begin{pmatrix} x_1, \ldots, x_{\alpha-1}, x, x_{\alpha+1}, \ldots, x_q \\ y_1, \ldots, y_{\beta-1}, y_\beta, y_{\beta+1}, \ldots, y_q \end{pmatrix}\lambda$$
$$+\lambda \int_a^b K(t, y) D\begin{pmatrix} x, x_1, \ldots, x_q \\ t, y_1, \ldots, y_q \end{pmatrix}\lambda\, dt.$$

Put $x_1 = x_1'$, ... , $x_q = x_q'$, $y_1 = y_1'$, ... , $y_q = y_q'$, $y = t$, $\lambda = \lambda_o$, divide by

$$D\begin{pmatrix} x_1', & \ldots, & x_q' \\ y_1', & \ldots, & y_q' \end{pmatrix}\lambda_o$$

and put

$$(40) \qquad H(x, y) = \frac{D\begin{pmatrix} x, x_1', & \ldots, & x_q' \\ y, y_1', & \ldots, & y_q' \end{pmatrix}\lambda_o}{D\begin{pmatrix} x_1', & \ldots, & x_q' \\ y_1', & \ldots, & y_q' \end{pmatrix}\lambda_o},$$

then we obtain

$$(41) \qquad \sum_{\alpha=1}^{q} \lambda_o K(x_\alpha', t)\varphi_\alpha(x) = \lambda_o K(x, t) - H(x, t) + \lambda_o \int_a^b H(x, s)K(s, t)ds.$$

The equation (39) can now be written

$$v(x) = \lambda_o \sum_{\alpha=1}^{q} \varphi_\alpha(x) \int_a^b K(x_\alpha', t)v(t)dt.$$

This shows that $v(x)$ can be written in the form (38) by taking for the constants C_α the values

$$C_\alpha = \lambda_o \int_a^b K(x_\alpha', t)v(t)dt.$$

Thus we obtain *Fredholm's second fundamental theorem:*

Theorem VIII.—*If* $\lambda = \lambda_o$ *is a root of* $D(\lambda) = 0$ *of order* q, *then the homogeneous integral equation*

$$(26) \qquad u(x) = \lambda_o \int_a^b K(x, t)u(t)dt$$

has q *linearly independent solutions in terms of which every other solution is expressible linearly and homogeneously. Such a system of* q *independent solutions is given by*

$$\varphi_\alpha(x) = \frac{D\begin{pmatrix} x_1', & \ldots, & x'_{\alpha-1}, & x, & x'_{\alpha+1}, & \ldots, & x_q' \\ y_1', & \ldots, & y'_{\beta-1}, & y_\beta', & y'_{\beta+1}, & \ldots, & y_q' \end{pmatrix}\lambda_o}{D\begin{pmatrix} x_1', & \ldots, & x_q' \\ y_1', & \ldots, & y_q' \end{pmatrix}\lambda_o}$$

$$(\alpha = 1, \ldots, q).$$

22. Characteristic Constant. Fundamental Functions. Definitions.—If $D(\lambda)$ is the Fredholm's determinant for the kernel $K(x, t)$ and $D(\lambda_o) = 0$, then λ_o is said to be a *characteristic constant* of the kernel $K(x, t)$. Further, if $\varphi(x)$ is continuous and not identically zero on the interval (ab) and

$$\varphi(x) = \lambda_o \int_a^b K(x, t)\varphi(t)dt,$$

then $\varphi(x)$ is called a *fundamental function* of the kernel $K(x, t)$, belonging to the characteristic constant λ_o.

$\varphi_1(x), \ . \ . \ . \ , \varphi_q(x)$ form a *complete system* of fundamental functions of the kernel $K(x, t)$, belonging to λ_o, if every other solution is expressible linearly in terms of these q solutions. Thus, if $\psi_1, \ . \ . \ . \ , \psi_q$ are any other q solutions, we must have

$$\psi_1 = C_{11}\varphi_1 + \ . \ . \ . \ + C_{1q}\varphi_q$$
$$. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$
$$\psi_q = C_{q1}\varphi_1 + \ . \ . \ . \ + C_{qq}\varphi_q$$

and if

$$\begin{vmatrix} C_{11} & \cdots & C_{1q} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ C_{q1} & \cdots & C_{qq} \end{vmatrix} \neq 0$$

$\psi_1, \ . \ . \ . \ , \psi_q$ form again a complete system of fundamental functions belonging to λ_o.

23. The Associated Homogeneous Integral Equation.—Preparatory to the discussion of the non-homogeneous integral equation, for $D(\lambda) = 0$, we will discuss the homogeneous integral equation

(42) $$v(x) = \lambda_o \int_a^b K(t, x)v(t)dt,$$

which is called the integral equation *associated* with the integral equation

(26) $$u(x) = \lambda_o \int_a^b K(x, t)u(t)dt.$$

Notice that the kernel

$$\overline{K}(x, t) \equiv K(t, x)$$

of the associated equation is derived from the original kernel $K(x, t)$ by interchanging the arguments x and t. There exist important relations between the solutions of the two equations (42) and (26). To obtain them, we first compute the Fredholm determinant and Fredholm minors for the kernel $\overline{K}(x, t)$, which we indicate by the corresponding dashed notation.

a) Fredholm's Determinant for the Associated Kernel.—

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n$$

where

$$A_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1, t_1) & \cdots & K(t_1, t_n) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ K(t_n, t_1) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1 \ldots dt_n.$$

Then

$$\overline{D}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \overline{A}_n$$

where

$$\overline{A}_n = \int_a^b \cdots \int_a^b \begin{vmatrix} (Kt_1, t_1) & \cdots & \overline{K}(t_1, t_n) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \overline{K}(t_n, t_1) & \cdots & \overline{K}(t_n, t_n) \end{vmatrix} dt_1 \ldots dt_n.$$

In this expression for $\bar{A}_n$ put $K(x, t) = K(t, x)$. Then

$$\bar{A}_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1, t_1) & \cdots & K(t_n, t_1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K(t_1, t_n) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1 \ldots dt_n.$$

The determinant which appears in the integrand of $\bar{A}_n$ is the same as that which appears in the integrand of the expression for A_n, with the exception that the rows and columns are interchanged. But this interchange leaves the value of the determinant unaltered. Therefore $\bar{A}_n = A_n$, and hence

(43) $$\bar{D}(\lambda) \equiv D(\lambda).$$

Hence also we conclude that $K(t, x)$ and $K(x, t)$ have the same characteristic constants.

b) Fredholm's Minors for the Associated Kernel.—We have from (30)

$$D\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix} \lambda\Bigg) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+p}}{n!} B_n\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix}$$

where the B_n are given by (28). Then

$$\bar{B}_n\begin{pmatrix} x_1, & \ldots & , x_p \\ y_1, & \ldots & , y_p \end{pmatrix} = \int_a^b \cdots \int_a^b$$

$$\begin{vmatrix} \bar{K}(x_1, y_1) & \cdots & \bar{K}(x_1, y_p) & \bar{K}(x_1, t_1) & \cdots & \bar{K}(x_1, t_n) \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \bar{K}(x_p, y_1) & \cdots & \bar{K}(x_p, y_p) & \bar{K}(x_p, t_1) & \cdots & \bar{K}(x_p, t_n) \\ \bar{K}(t_1, y_1) & \cdots & \bar{K}(t_1, y_p) & \bar{K}(t_1, t_1) & \cdots & \bar{K}(t_1, t_n) \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \bar{K}(t_n, y_1) & \cdots & \bar{K}(t_n, y_p) & \bar{K}(t_n, t_1) & \cdots & \bar{K}(t_n, t_n) \end{vmatrix} dt_1 \ldots dt_n.$$

But $\overline{K}(x, t) = K(t, x)$, then

$$\overline{B}_n\begin{pmatrix} x_1, & \cdots & , x_p \\ y_1, & \cdots & , y_p \end{pmatrix} = \int_a^b \cdots \int_a^b$$

$$\begin{vmatrix} K(y_1, x_1) & \cdots & K(y_p, x_1) & K(t_1, x_1) & \cdots & K(t_n, x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K(y_1, x_p) & \cdots & K(y_p, x_p) & K(t_1, x_p) & \cdots & K(t_n, x_p) \\ K(y_1, t_1) & \cdots & K(y_p, t_1) & K(t_1, t_1) & \cdots & K(t_n, t_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K(y_1, t_n) & \cdots & K(y_p, t_n) & K(t_1, t_n) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1 \ldots dt_n.$$

An interchange of rows and columns in the determinant of the integrand does not change the value of the determinant. Hence

$$\overline{B}_n\begin{pmatrix} x_1, & \cdots & , x_p \\ y_1, & \cdots & , y_p \end{pmatrix} = \int_a^b \cdots \int_a^b K\begin{pmatrix} y_1, & \cdots & , y_p, t_1, & \cdots & , t_n \\ x_1, & \cdots & , x_p, t_1, & \cdots & , t_n \end{pmatrix} dt_1 \ldots dt_n.$$

Then, by means of the equation (28) defining B_n, we see that

$$\overline{B}_n\begin{pmatrix} x_1, & \cdots & , x_p \\ y_1, & \cdots & , y_p \end{pmatrix} = B_n\begin{pmatrix} y_1, & \cdots & , y_p \\ x_1, & \cdots & , x_p \end{pmatrix}$$

and, therefore,

(44) $$\overline{D}\begin{pmatrix} x_1, & \cdots & , x_p \\ y_1, & \cdots & , y_p \end{pmatrix}\lambda \equiv D\begin{pmatrix} y_1, & \cdots & , y_p \\ x_1, & \cdots & , x_p \end{pmatrix}\lambda.$$

Since λ_o is a characteristic constant of $K(x, t)$ of index q, we have

$$D\begin{pmatrix} x_1, & \cdots & , x_p \\ y_1, & \cdots & , y_p \end{pmatrix}\lambda_o \equiv 0 \qquad (p = 1, \ldots, q - 1)$$

while

$$D\begin{pmatrix} x_1', & \cdots & , x_q' \\ y_1', & \cdots & , y_q' \end{pmatrix}\lambda_o \neq 0.$$

Hence, by (44)

$$\bar{D}\begin{pmatrix} x_1, & . & . & . & , x_p \\ y_1, & . & . & . & , y_p \end{pmatrix}\lambda_o\Big) = D\begin{pmatrix} y_1, & . & . & . & , y_p \\ x_1, & . & . & . & , x_p \end{pmatrix}\lambda_o\Big) = 0$$

in $x_1, \ldots, x_p, y_1, \ldots y_p$ for $p = 1, \ldots, q - 1$.

Further, if we put

(45) $$\bar{x}_\alpha{}' = y_\alpha{}', \quad \bar{y}_\alpha{}' = x_\alpha{}',$$

we have

$$\bar{D}\begin{pmatrix} \bar{x}_1{}', & . & . & . & , \bar{x}_q{}' \\ \bar{y}_1{}', & . & . & . & , \bar{y}_q{}' \end{pmatrix}\lambda_o\Big) = D\begin{pmatrix} \bar{y}_1{}', & . & . & . & , \bar{y}_q{}' \\ \bar{x}_1{}', & . & . & . & , \bar{x}_q{}' \end{pmatrix}\lambda_o\Big)$$
$$= D\begin{pmatrix} x_1{}', & . & . & . & , x_q{}' \\ y_1{}', & . & . & . & , y_q{}' \end{pmatrix}\lambda_o\Big) \neq 0.$$

Hence, by definition, the index $\bar{q}$ of λ_o as a characteristic constant of $K(t, x)$ is q. We state this result in the

Theorem IX.—*If λ_o is a characteristic constant of $K(x, t)$ of index q, then λ_o is a characteristic constant of $K(t, x)$ of the same index:*

$$\bar{q} = q.$$

c) *The Fundamental Functions of the Associated Equation.* Apply, now, Theorem VIII to equation (42) and we find that it has q linearly independent solutions. A fundamental system of such solutions is given by

$$\bar{\varphi}_\alpha(x) = \frac{\bar{D}\begin{pmatrix} \bar{x}_1{}', & . & . & . & , \bar{x}'_{\alpha-1}, & x, & \bar{x}'_{\alpha+1}, & . & . & . & , \bar{x}_q{}' \\ \bar{y}_1{}', & . & . & . & , \bar{y}'_{\alpha-1}, & \bar{y}_\alpha{}', & \bar{y}'_{\alpha+1}, & . & . & . & , \bar{y}_q{}' \end{pmatrix}\lambda_o\Big)}{\bar{D}\begin{pmatrix} \bar{x}_1{}', & . & . & . & , \bar{x}_q{}' \\ \bar{y}_1{}', & . & . & . & , \bar{y}_q{}' \end{pmatrix}\lambda_o\Big)}$$

$$= \frac{D\begin{pmatrix} \bar{y}_1{}', & . & . & . & , \bar{y}'_{\alpha-1}, & \bar{y}_\alpha{}', & \bar{y}'_{\alpha+1}, & . & . & . & , \bar{y}_q{}', \\ \bar{x}_1{}', & . & . & . & , \bar{x}'_{\alpha-1}, & x, & \bar{x}'_{\alpha+1}, & . & . & . & , \bar{x}_q{}' \end{pmatrix}\lambda_o\Big)}{D\begin{pmatrix} \bar{y}_1{}', & . & . & . & , \bar{y}_q{}' \\ \bar{x}_1{}', & . & . & . & , \bar{x}_q{}' \end{pmatrix}\lambda_o\Big)}.$$

If, now, we introduce the change of notation (45), we find

$$\bar{\varphi}_\alpha(x) = \frac{D\begin{pmatrix} x_1', & \ldots, & x'_{\alpha-1}, & x_\alpha', & x'_{\alpha+1}, & \ldots, & x_q' \\ y_1', & \ldots, & y'_{\alpha-1}, & x\ , & y'_{\alpha+1}, & \ldots, & y_q' \end{pmatrix}\lambda_o}{D\begin{pmatrix} x_1', & \ldots, & x_q' \\ y_1', & \ldots, & y_q' \end{pmatrix}\lambda_o}$$

$$(\varphi = 1, \ldots, q).$$

The most general solution of (42) is now

$$v(x) = C_1\bar{\varphi}_1(x) + C_2\bar{\varphi}_2(x) + \ldots + C_q\bar{\varphi}_q(x).$$

d) The Function $H(x, y)$ for the Associated Kernel.—From the definition of $H(x, y)$ given earlier, we have

$$\bar{H}(x, y) = \frac{\bar{D}\begin{pmatrix} x, \bar{x}_1', & \ldots, & \bar{x}_q' \\ y, \bar{y}_1', & \ldots, & \bar{y}_q' \end{pmatrix}\lambda_o}{\bar{D}\begin{pmatrix} \bar{x}_1', & \ldots, & \bar{x}_q' \\ \bar{y}_1', & \ldots, & \bar{y}_q' \end{pmatrix}\lambda_o}.$$

Apply (44) and make the change of notation (45). Then

$$(46)\qquad \bar{H}(x, y) = \frac{D\begin{pmatrix} y, y_1', & \ldots, & y_q' \\ x, x_1', & \ldots, & x_q' \end{pmatrix}\lambda_o}{D\begin{pmatrix} y_1', & \ldots, & y_q' \\ x_1', & \ldots, & x_q' \end{pmatrix}\lambda_o} = H(y, x).$$

If, now, we take account of (46) and make the change of notation (45), then the relation (41) written for the kernel $\bar{K}(x, t) = K(t, x)$ becomes

$$(47)\quad \sum_{\alpha=1}^{q} \lambda_o K(t, y_\alpha')\bar{\varphi}_\alpha(x) = \lambda_o K(t, x) - H(t, x) + \lambda_o \int_a^b H(s, x)K(t, s)ds.$$

e) The Orthogonality Theorem.—

Theorem X.—*If λ_o and λ_1 are two distinct characteristic constants of $K(x, t)$, $\varphi_o(x)$ is a fundamental function of*

$K(x, t)$ for λ_o, and $\bar{\varphi}_1(x)$ is a fundamental function of $\bar{K}(x, t)$ for λ_1, that is

$$(48)\quad \varphi_o(x) = \lambda_o \int_a^b K(x, t)\varphi_o(t)dt$$

$$(49)\quad \bar{\varphi}_1(x) = \lambda_1 \int_a^b \bar{K}(x, t)\bar{\varphi}_1(t)dt = \lambda_1 \int_a^b K(t, x)\bar{\varphi}_1(t)dt$$

then

$$(50)\qquad \int_a^b \varphi_o(x)\bar{\varphi}_1(x) = 0.$$

Proof.—From (48) and (49) we obtain

$$(\lambda_o - \lambda_1)\int_a^b \varphi_o(x)\bar{\varphi}_1(x)dx = \lambda_o\lambda_1 \int_a^b\int_a^b \varphi_o(x)K(t, x)\bar{\varphi}_1(t)\, dt\, dx - \lambda_o\lambda_1 \int_a^b\int_a^b \bar{\varphi}_1(x)K(x, t)\varphi_o(t)\, dt\, dx.$$

We see that the two integrals on the right are equal if in the last integral we write t and x in place of x and t. Then, since by hypothesis $\lambda_o \neq \lambda_1$, we must have (50).

Definition.—Two continuous functions $g(x)$, $h(x)$, for which

$$\int_a^b g(x)h(x)dx = 0,$$

are said to be *orthogonal* to each other.

Hence the above result may be stated as follows: $\varphi_o(x)$ and $\bar{\varphi}_1(x)$ are orthogonal to each other.

24. The Non-homogeneous Integral Equation When D(λ) = 0.—With the aid of the results established in the last article we can discuss completely the solution of the non-homogeneous integral equation

$$(51)\qquad u(x) = f(x) + \lambda_o \int_a^b K(x, t)u(t)dt$$

when $D(\lambda_o) = 0$ and λ_o is of index q.

The finite system of linear equations

$$u_i - \lambda h \sum_{j=1}^{n} K_{ij} u_j = f_i \qquad (i = 1, \ . \ . \ . \ , n),$$

of which the equation (51) may be considered as a limit, has, in general, for $\Delta = 0$, no finite solution. If, however, certain conditions on the f_i are satisfied, the system has an infinitude of solutions. In analogy we will find that (51), for $D(\lambda_o) = 0$, has, in general, no solutions. If, however, $f(x)$ satisfies certain conditions, then (51) has an infinitude of solutions.

a) Necessary Conditions.—To obtain these conditions $f(x)$ we assume that $u(x)$ is a continuous function of x satisfying (51). Multiply both sides of (51) by $\bar{\varphi}_\alpha(x)$, where

$$\bar{\varphi}_\alpha(x) = \lambda_o \int_a^b K(t, x)\bar{\varphi}_\alpha(t)dt,$$

and integrate with respect to x from a to b. We obtain

$$(52) \quad \int_a^b f(x)\bar{\varphi}_\alpha(x)dx = \int_a^b u(x)\bar{\varphi}_\alpha(x)dx - \lambda_o \int_a^b \bar{\varphi}_\alpha(x)\left\{\int_a^b K(x, t)u(t)dt\right\}dx.$$

In the last integral on the right, $\bar{\varphi}_\alpha(x)$ is constant with respect to t and so can be placed under the second sign of integration. We may change the order of integration and then take $u(t)$ from under the sign of integration with respect to x. Thus the last term becomes

$$\int_a^b u(t)\left\{\lambda_o \int_a^b \bar{\varphi}_\alpha(x)K(x, t)dx\right\}dt = \int_a^b u(t)\bar{\varphi}_\alpha(t)dt.$$

Thus we see that the first and last terms on the right cancel and

$$(53) \qquad \int_a^b f(x)\bar{\varphi}_\alpha(x)dx = 0. \qquad (\alpha = 1, \ . \ . \ . \ , q)$$

Hence, in order that there may exist a continuous solution $u(x)$ of (51), $f(x)$ must satisfy the q conditions (53).

b) Sufficiency Proof.—Let us now show conversely that, if $f(x)$ satisfies the q conditions (53), then (51) does have a solution. By our hypothesis the q equations (53) are satisfied. Then

$$\sum_{\alpha=1}^{q} \lambda_o K(x, y_\alpha') \int_a^b f(t)\bar{\varphi}_\alpha(t)dt = 0.$$

Now, $\lambda_o K(x, y_\alpha')$ is independent of t and so may be placed under the sign of integration. It is permissible to change the order of performing the summation and the integration. We thus obtain

$$\int_a^b \left\{ \sum_{\alpha=1}^{q} \lambda_o K(x, y_\alpha')\bar{\varphi}_\alpha(t)f(t) \right\} dt = 0,$$

which, on account of (47), becomes

$$(54) \quad 0 = \int_a^b \lambda_o K(x, t)f(t)dt - \int_a^b H(x, t)f(t)dt + \lambda_o \int_a^b f(t) \left\{ \int_a^b H(s, t)K(x, s)ds \right\} dt.$$

The last term may be written

$$\lambda_o \int_a^b \int_a^b f(t)H(s, t)K(x, s)ds\, dt.$$

Make now a change in notation. In place of t and s write s and t. We obtain

$$\lambda_o \int_a^b \int_a^b f(s)H(t, s)K(x, t)dt\, ds.$$

In this definite double integral it is permissible to change the order of integration. We then obtain

$$\lambda_o \int_a^b K(x, t) \left\{ \int H(t, s)f(s)ds \right\} dt.$$

After making these reductions in (54), combine the first and last terms and obtain

$$(55)\quad 0 = \lambda_o \int_a^b K(x, t)\left\{f(t) + \int_a^b H(t, s)f(s)ds\right\}dt - \int_a^b H(x, t)f(t)dt$$

Now put

$$u_o(t) = f(t) + \int_a^b H(t, s)f(s)ds,$$

then

$$\int_a^b H(x, t)f(t)dt = u_o(x) - f(x).$$

Making use of these last two equations, (55) becomes

$$u_o(x) = f(x) + \lambda_o \int_a^b K(x, t)u_o(t)dt.$$

Thus we have proved that if (53) are satisfied then (51) has at least one solution, $u_o(x)$ given by

$$(56)\qquad u_o(x) = f(x) + \int_a^b H(x, t)f(t)dt.$$

c) Determination of All Solutions.—Let us suppose that (51) has another continuous solution $u(x)$. Then $u(x) - u_o(x)$ is a solution of the homogeneous equation

$$(57)\qquad v(x) = \lambda_o \int_a^b K(x, t)v(t)dt,$$

for, if we subtract the members of (56) from the corresponding members of (51), we obtain

$$(58)\quad u(x) - u_o(x) = \lambda_o \int_a^b K(x, t)\Big[u(t) - u_o(t)\Big]dt.$$

By Theorem VIII the most general solution of (57) is of the form

$$C_1\varphi_1(x) + C_2\varphi_2(x) + \ . \ . \ . \ + C_q\varphi_q(x),$$

where $C_1, C_2, \ . \ . \ . \ , C_q$ are arbitrary constants.

Hence the most general solution of (58) is

$$u(x) - u_o(x) = C_1\varphi_1(x) + C_2\varphi_2(x) + \ . \ . \ . \ + C_q\varphi_q(x).$$

Therefore,

$$u(x) = f(x) + \int_a^b H(x,t)f(t)dt + C_1\varphi_1(x) + \cdots + C_q\varphi_q(x)$$

is the complete solution of (51). We have thus proved *Fredholm's third fundamental theorem:*

Theorem XI.—*If λ_o is a characteristic constant of $K(x, t)$ of index q then*

$$(51) \qquad u(x) = f(x) + \lambda_o \int_a^b K(x, t)u(t)dt$$

has, in general, no continuous solution. In order that a continuous solution exist it is necessary that

$$\int_a^b f(x)\bar{\varphi}_\alpha(x)dx = 0, \quad \alpha = 1, \ldots, q$$

where the $\bar{\varphi}_\alpha(x)$ are a complete set of fundamental functions for the associated homogeneous equation

$$(42) \qquad v(x) = \lambda_o \int_a^b K(t, x)v(t)dt.$$

If these conditions are satisfied, then there are a q-fold infinitude of solutions of (51) given by

$$u(x) = f(x) + \int_a^b H(x, t)f(t)dt + C_1\varphi_1(x) + \cdots + C_q\varphi_q(x)$$

where $C_1, \ldots, C_q$ are arbitrary constants and where the $\varphi_\alpha(x)$ are a complete set of fundamental functions for

$$(26) \qquad u(x) = \lambda_o \int_a^b K(x, t)u(t)dt \quad \text{(Theorem VIII)}$$

and $H(x, t)$ is given by (40).

The following table exhibits the results of the solution of Fredholm's equation together with the analogy between the finite system and the integral equation.

$$u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$$

Case I: $D(\lambda) \neq 0$		Case II: $D(\lambda) = 0$, index q	
Non-homogeneous	Homogeneous	Non-homogeneous	Homogeneous
Unique solution $u(x) = f(x) + \int_a^b \frac{D(x, t; \lambda)}{D(\lambda)} f(t)dt$	Unique solution $u \equiv 0$	In general, no solution Solutions exist only if f satisfies $\int_a^b f(x)\bar{\varphi}_\alpha(x) = 0$ Then ∞^q solutions	∞^q solutions: $\sum_{\alpha=1}^{q} C_\alpha \varphi_\alpha(x)$

$$u_i - \lambda h \sum_{j=1}^{n} K_{ij} u_j = f_i, \; i = 1, \ldots, n$$

Case I: $\Delta \neq 0$		Case II: $\Delta = 0$, index q	
Non-homogeneous	Homogeneous	Non-homogeneous	Homogeneous
Unique solution $u_k = \frac{\sum_{i=1}^{n} f_i \Delta_{ik}}{\Delta}$	Unique solution $u_k = 0$	In general, no solution ∞^q solutions if the f_i satisfy certain q relations of the form $C_{\alpha 1} f_1 + \ldots + C_{\alpha n} f_n = 0,$ $\alpha = 1, \ldots, q$	∞^q solutions if Δ is of rank r where $q = n - r$

25. Kernels of the Form $\sum a_i(x)b_i(y)$. We give a brief discussion of the integral equation with a kernel of the form

$$K(x, y) = a_1(x)b_1(y) + \cdots + a_n(x)b_n(y).$$

Fredholm's integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$$

can now be written in the form

$$(59)\quad u(x) = f(x) + \lambda\left[a_1(x)\int_a^b b_1(t)u(t)dt + \cdots + a_n(x)\int_a^b b_n(t)u(t)dt\right].$$

If, now, we put

$$(60)\quad \int_a^b b_i(t)u(t)dt = K_i \qquad (i = 1, \ldots, n),$$

we see that $u(x)$ is of the form

$$(61)\quad u(x) = f(x) + \lambda\left[a_1(x)K_1 + \cdots + a_n(x)K_n\right].$$

In order to determine the constants K_i, let us substitute in (60) the value of u given by (61). We obtain the n equations

$$(62)\quad K_i - \lambda\left[\int_a^b a_1(t)b_i(t)K_1dt + \cdots + \int_a^b a_n(t)b_i(t)K_ndt\right] = \int_a^b b_i(t)f(t)dt \quad (i = 1, \ldots, n).$$

Introducing the notation

$$\int_a^b a_k(t)b_i(t)dt = C_{ki}$$

the system (62) can be written in the form

$$(63)\quad K_i - \lambda\left[C_{1i}K_1 + \cdots + C_{ni}K_n\right] = \int_a^b b_i(t)f(t)dt.$$

Equations (63) are a linear algebraic system of n non-homogeneous equations in the n unknowns $K_1, \ldots, K_n$, with the determinant

$$D(\lambda) = \begin{vmatrix} 1 - \lambda C_{11} & -\lambda C_{12} & \cdots & -\lambda C_{1n} \\ -\lambda C_{21} & 1 - \lambda C_{22} & \cdots & -\lambda C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda C_{n1} & -\lambda C_{n2} & \cdots & 1 - \lambda C_{nn} \end{vmatrix}$$

From the theory of linear systems, we have at once the result:

a) If $D(\lambda) \neq 0$, *the system* (62) *is satisfied by one and only one set of values of* $K_1, \ldots, K_n$ *and these values are given by Cramer's formulas. Therefore, Fredholm's equation* (59) *has one and only one solution, which is given by* (61).

b) If $D(\lambda) = 0$ for $\lambda = \lambda_o$ (and this happens for n values of λ, real or complex), and one of the qth minors of $D(\lambda)$ is the first minor which does not vanish for $\lambda = \lambda_o$ (this qth minor is a determinant of order $n - q$), then the general solution of the homogeneous system (62) ($f(x) \equiv 0$) will be of the form[1]

$$K_i = \alpha_1 m_{1i} + \alpha_2 m_{2i} + \cdots + \alpha_q m_{qi} \quad (i = 1, \ldots, n),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_q$ are arbitrary constants.

If we put the values of K_i so obtained in (61), we obtain

$$u(x) = \lambda_o \left[\alpha_1 u_1(x) + \alpha_2 u_2(x) + \cdots + \alpha_q u_q(x) \right]$$

where the functions

$$u_r(x) = m_{r1} a_1(x) + m_{r2} a_2(x) + \cdots + m_{rn} a_n(x) \qquad (r = 1, \ldots, q)$$

are linearly independent.

Thus we see that, under the circumstances specified, the homogeneous integral equation for $\lambda = \lambda_o$ has q linearly independent solutions.

[1] BÖCHER, "Introduction to Higher Algebra," §18.

The *associated equation*

$$(64)\quad \bar{u}(x) = f(x) + \lambda\left[b_1(x)\int_a^b a_1(t)\bar{u}(t)dt + \cdots + b_n(x)\int_a^b a_n(t)\bar{u}(t)dt\right]$$

is obtained from (59) by interchanging the functions $a_i(x)$ and $b_i(x)$. The general term of the characteristic determinant of (59) being

$$\int_a^b a_k(t)b_i(t)dt = C_{ki},$$

the general term for the associated equation will be

$$\int_a^b b_k(t)a_i(t)dt = C_{ik}.$$

The characteristic determinants of these two equations are identical, since one can obtain one from the other by interchanging rows and columns. Therefore, the equation (59) and the associated equation (64) have exactly the same characteristic numbers and with the same index.

From the general theory we know that if λ_o is a root of $D(\lambda) = 0$ of index q, then, in order that the non-homogeneous equation (59) may have a solution, we must have

$$\int_a^b f(x)\bar{u}_i(x)\,dx = 0\ (i = 1, \ldots, q).$$

EXERCISES

For the equation $u(x) = f(x) + \lambda\int_a^b K(x, t)u(t)dt$ compute $D(\lambda)$ and $D(x, y; \lambda)$ for the following kernels for the specified limits a and b:

	Ans.
1. $K(x, t) = 1, a = 0, b = 1.$	$D(\lambda) = 1 - \lambda.$
2. $K(x, t) = -1, a = 0, b = 1.$	$D(\lambda) = 1 + \lambda.$
3. $K(x, t) = \sin x, a = 0, b = \pi.$	$D(\lambda) = 1 - 2\lambda.$
4. $K(x, t) = xt, a = 0, b = 10.$	$D(\lambda) = 1 - \frac{1{,}000}{3}\lambda.$

5. $K(x, t) = t$, $a = 0$, $b = 10$. $D(\lambda) = 1 - 50\lambda$.
6. $K(x, t) = x$, $a = 4$, $b = 10$. $D(\lambda) = 1 - 42\lambda$.
7. $K(x, t) = g(x)$, $a = a$, $b = b$. $D(\lambda) = 1 - \lambda \int_a^b g(t)dt$.
8. $K(x, t) = g(t)$, $a = a$, $b = b$. $D(\lambda) = 1 - \lambda \int_a^b g(t)dt$.
9. $K(x, t) = 2e^x \cdot e^t$, $a = 0$, $b = 1$. $D(\lambda) = 1 - (e^2 - 1)\lambda$.
10. $K(x, t) = x - t$, $a = 0$, $b = 1$.

Solve the following integral equations:

11. $u(x) = \sec^2 x + \lambda \int_0^1 u(t)dt.$

12. $u(x) = \sec x \cdot \tan x - \lambda \int_0^1 u(t)dt.$

13. $u(x) = \cos x + \lambda \int_0^\pi \sin x \cdot u(t)dt.$

14. $u(x) = e^x + \lambda \int_0^{10} xt \cdot u(t)dt.$

15. $u(x) = x^2 + \lambda \int_0^{10} t \cdot u(t)dt.$

16. $u(x) = \sin x + \lambda \int_4^{10} x \cdot u(t)dt.$

17. $u(x) = e^x + \lambda \int_0^1 2e^x e^t u(t)dt.$

Solve the following homogeneous integral equations:

18. $u(x) = \int_0^1 u(t)dt.$

19. $u(x) = \int_0^1 (-1)u(t)dt.$

20. $u(x) = \frac{1}{2}\int_0^\pi \sin x . u(t)dt.$

21. $u(x) = \frac{3}{1{,}000}\int_0^{10} xt . u(t)dt.$

22. $u(x) = \frac{1}{50}\int_0^{10} t . u(t)dt.$

23. $u(x) = \frac{1}{42}\int_4^{10} x.u(t)dt.$

24. $u(x) = \frac{1}{e^2 - 1}\int_0^1 2e^x e^t u(t)dt.$

Solve the following equation by the method of §25:

25. $u(x) = x^2 + \lambda\int_0^1 (1 + xt)u(t)dt.$

26. $u(x) = x + \lambda\int_0^\pi (1 + \sin x \sin t)u(t)dt.$

27. $u(x) = x + \lambda\int_0^1 (1 + x + t)u(t)dt.$

28. $u(x) = x + \lambda\int_0^1 (x - t)u(t)dt.$

29. $u(x) = x + \lambda\int_0^1 (x - t)^2 u(t)dt.$

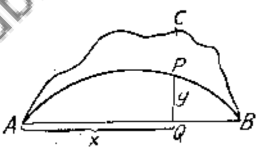
30. Solve Exercises 11–17 inclusive by this method.

CHAPTER IV

APPLICATIONS OF THE FREDHOLM THEORY

I. Free Vibrations of an Elastic String

26. The Differential Equations of the Problem.—We consider an elastic string stretched between the two fixed points A and B. We pull it out of its position of equilibrium AQB into some other plane initial position, as ACB, and then release it. The string will describe transverse vibrations. Suppose that at time t the string occupies the position APB. Let $x = AQ$, $y = QP$ be the abscissa and ordinate of any one of its points P. Then y is a function of x and t. We suppose the cross-section of the string to be constant and infinitesimal compared with the length.



[Fig. 5.

The string is of homogeneous density. The effect of gravity is to be neglected. Further, we take for simplicity $AB = 1$. It is then proved in the theory of elasticity that the motion of the string is given by the partial differential equation

$$(1) \qquad \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad (c = \text{constant})$$

with the initial conditions

$$(2) \qquad y(0, t) = 0, \qquad\qquad y(1, t) = 0,$$

$$(3) \qquad y(x, 0) = g(x), \qquad\qquad y_t(x, 0) = 0,$$

where $y = g(x)$ is the equation of the initial position ACB. Equations (2) are the analytic statement of the fact that the end points A and B remain fixed, while equations (3) state that for $t = 0$ the string is in the initial position ACB and each particle starts its motion with an initial zero velocity.[1]

27. Reduction to a One-dimensional Boundary Problem. Let us try to find a solution of (1) in the form

$$y = u(x)\varphi(t).$$

Substitution of this value of y in (1) gives

$$u(x)\frac{d^2\varphi}{dt^2} = c^2\varphi(t)\frac{d^2u}{dx^2},$$

which can be put in the form

$$\frac{\dfrac{d^2\varphi(t)}{dt^2}}{\varphi(t)} = c^2\frac{\dfrac{d^2u(x)}{dx^2}}{u(x)}.$$

The right-hand side is independent of t, and the left-hand side is independent of x. Then either member is a constant, which we designate as $-\lambda c^2$. This gives us the two ordinary differential equations to solve:

$$\frac{d^2u}{dx^2} + \lambda u = 0, \qquad \frac{d^2\varphi}{dt^2} + \lambda c^2\varphi = 0.$$

The initial conditions on u and φ are obtained from (2) and (3), and for u are as follows:

$$\begin{matrix} u(0)\varphi(t) = 0 \\ u(1)\varphi(t) = 0 \end{matrix} \quad \text{whence} \quad \begin{matrix} u(0) = 0 \\ u(1) = 0 \end{matrix} \text{ since } \varphi(t) \not\equiv 0.$$

We are thus led to the following *one-dimensional boundary problem*: to determine a function $u(x)$ which will satisfy the differential equation

(4) $$\frac{d^2u}{dx^2} + \lambda u = 0$$

and the boundary conditions

$$u(0) = 0, \qquad u(1) = 0.$$

28. Solution of the Boundary Problem.—We see at once that $u \equiv 0$ is a solution. This gives for the original prob-

[1] WEBER-RIEMANN, "Lehrbuch der Partielle Differentialgleichungen," §83.

lem of the vibrating string the trivial solution $y \equiv 0$, which means that the string remains at rest. Hereafter, when we refer to a solution of our boundary problem we shall mean a solution not identically zero. To obtain the solution we have three cases to consider.

Case I.—$\lambda > 0$. From the elementary theory of differential equations we know that the most general solution of (4) is

$$u(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}\cdot x.$$

The conditions $u(0) = 0$, $u(1) = 0$ give us $A = 0$ and either (1) $B = 0$, or (2) $\sin \sqrt{\lambda} = 0$.

(1) If $B = 0$, we obtain the trivial solution $u \equiv 0$.

(2) If $\sin \sqrt{\lambda} = 0$, then $\lambda = n^2\pi^2$ (n, an integer) and the solution is

$$u(x) = B \sin n\pi x$$

Case II.—$\lambda = 0$. The general solution of (4) is now

$$u = Ax + B.$$

But $u(0) = B = 0$, and $u(1) = A = 0$, so that we have again the trivial solution $u \equiv 0$.

Case III.—$\lambda < 0$. The general solution of (4) is now

$$u = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

Applying the initial conditions, we obtain

$$u(0) = A + B = 0, \text{ whence } B = -A$$
$$u(1) = A(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}) = 0, \text{ whence } A = 0\ (\lambda \neq 0).$$

Therefore, $A = B = 0$, and we have again the trivial solution $u \equiv 0$.

Thus we arrive at the

Theorem I.—*If $\lambda = n^2\pi^2$(n, an integer), then the boundary problem*

$$\text{(4)} \qquad \frac{d^2u}{dx^2} + \lambda u = 0,\ u(0) = 0,\ u(1) = 0$$

has an infinitude of solutions:

$$u = B \sin n\pi x.$$

If $\lambda \neq n^2\pi^2$, then the only solution is the trivial one $u \equiv 0$.

29. Construction of Green's Function.—We now propose to show that every solution of our boundary problem (4) satisfies at the same time a linear integral equation. We observe first that with the given boundary conditions the method of §4 cannot be used to determine an equivalent integral equation. In the present instance, in order to determine an equivalent integral equation, we first construct the *Green's function* belonging to the boundary problem.

The given boundary problem for $\lambda = 0$ has only the trivial solution $u \equiv 0$. This is true, however, only under the assumptions tacitly made throughout, namely, that u, together with its first and second derivatives, is continuous in the interval [01]. Let us use the notation $u^{c''}$ to denote this assumption.[1]

Drop now the assumption $u^{c''}$ and allow the derivative u' to be discontinuous at an arbitrarily prescribed point ξ between 0 and 1, while u itself remains continuous. Accordingly, we propose to determine a function u satisfying the following conditions:

A) u^c in [01].

B) $u^{c''}$ and $\dfrac{d^2u}{dx^2} = 0$ in $[0\xi]$.

$u^{c''}$ and $\dfrac{d^2u}{dx^2} = 0$ in $[\xi 1]$.

C) $u(0) = 0, u(1) = 0.$

The solution then must be of the form

$$u = \begin{cases} \alpha_0 x + \beta_0 \text{ in } [0\xi] \\ \alpha_1 x + \beta_1 \text{ in } [\xi 1]. \end{cases}$$

[1] Similarly, let us use the notation C to denote the class of all continuous functions and the notation $C^{(n)}$ to denote the class of all functions having continuous derivatives up to the order n inclusive.

From C) we find

$$u(0) = \beta_0 = 0; \; u(1) = \alpha_1 + \beta_1 = 0.$$

Thus the solution reduces to

$$u = \begin{cases} \alpha_0 x & [0\xi] \\ \beta_1(1 - x) & [\xi 1]. \end{cases}$$

The condition A) must be satisfied, whence

$$\alpha_0 \xi = \beta_1(1 - \xi).$$

Therefore $\alpha_0 = \rho(1 - \xi)$ and $\beta_1 = \rho\xi$.

The solution now takes the form

$$u = \begin{cases} \rho(1 - \xi)x & [0\xi] \\ \rho\xi(1 - x) & [\xi 1]. \end{cases}$$

Geometrically, this solution is represented by a broken line as in the adjoining figure.

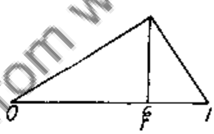


FIG. 6.

For $x = \xi$, the derivative has a discontinuity measured by

$$u'(\xi - 0) - u'(\xi + 0) = \rho(1 - \xi) + \rho\xi = \rho.$$

We now impose the further condition that this discontinuity shall be $+ 1$. Then $\rho = 1$. The function u so obtained is called *Green's function* for the boundary problem (4). We use the notation $K(x, \xi)$ to represent this function. We have thus the following:

Theorem II.—*There exists one and only one function $K(x, \xi)$ which satisfies the conditions*

$A)$ K^c in [01].

$B)$ $K^{c\prime\prime}$ and $\dfrac{d^2K}{dx^2} = 0$ in $[0\xi]$.

$K^{c\prime\prime}$ and $\dfrac{d^2K}{dx^2} = 0$ in $[\xi 1]$.

$C)$ $K(0) = 0,\ K(1) = 0.$

$D)$ $\Big[K'\Big]_{x = \xi + 0}^{x = \xi - 0} = +1, \quad 0 < \xi < 1.$

This function is given by the formula

(5)
$$K(x, \xi) = \begin{cases} (1 - \xi)x \text{ for } 0 \leqq x \leqq \xi \\ \xi(1 - x) \text{ for } \xi \leqq x \leqq 1. \end{cases}$$

Let us use the notation

$$K_0(x, \xi) = (1 - \xi)x$$
$$K_1(x, \xi) = \xi(1 - x).$$

The properties $A)$ and $D)$ of $K(x, \xi)$ may now be written

$$K_0(\xi, \xi) = K_1(\xi, \xi)$$
$$K_0'(\xi, \xi) - K_1'(\xi, \xi) = 1.$$

We can now prove the following:

Theorem III.—*Green's function is symmetric in x and ξ:*

$$K(x, \xi) = K(\xi, x).$$

Proof.—Let $0 \leqq z_1 \leqq z_2 \leqq 1$.

Then $\quad K(z_1, z_2) = K_0(z_1, z_2) = (1 - z_2)z_1$

while $\quad K(z_2, z_1) = K_1(z_1, z_2) = z_1(1 - z_2)$

whence $\quad K(z_1, z_2) = K(z_2, z_1).$

30. Equivalence between the Boundary Problem and a Linear Integral Equation.—Take the equations

$$\frac{d^2u}{dx^2} = -\lambda u, \ \frac{d^2K}{dx^2} = 0.$$

Multiply the first by $-K$ and the second by u and add. We obtain

$$uK'' - Ku'' = \lambda uK, \text{ or}$$
$$\frac{d}{dx}(uK' - Ku') = \lambda uK.$$

This equality holds in each of the two subintervals $[0\xi]$ and $[\xi 1]$. Integration over each of them gives

$$\left[uK' - Ku'\right]_0^{\xi-0} = \lambda \int_0^{\xi-0} uKdx$$
$$\left[uK' - Ku'\right]_{\xi+0}^1 = \lambda \int_{\xi+0}^1 uKdx.$$

Both u and K vanish at 0 and 1. u, u', and K are continuous functions of x over the whole interval [01]. Whence, adding the last two equations, we obtain

$$u(\xi)\left[K'(\xi - 0) - K'(\xi + 0)\right] = \lambda \int_0^1 K(x, \xi)u(x)dx.$$

But $K'(\xi - 0) - K'(\xi + 0) = 1$, and so

$$u(\xi) = \lambda \int_0^1 K(x, \xi)u(x)dx.$$

Now interchange x and ξ and remember that $K(x, \xi) = K(\xi, x)$. Then

$$u(x) = \lambda \int_0^1 K(x, \xi)u(\xi)d\xi.$$

This is a homogeneous linear integral equation of the second kind for the determination of $u(x)$. Every solution of the original boundary problem (4) satisfies this integral equation. Hence, we have the following:

Theorem IV.—*If $u(x)$ has continuous first and second derivatives, and satisfies the boundary problem*

(4) $$\frac{d^2u}{dx^2} + \lambda u = 0,\ u(0) = 0,\ u(1) = 0,$$

then $u(x)$ is continuous and satisfies the homogeneous linear integral equation

$$u(x) = \lambda \int_0^1 K(x, \xi)u(\xi)d\xi,$$

where $K(x, \xi)$ is given by the formula (5).

Let us now prove the following converse

Theorem V.—*If $u(x)$ is continuous and satisfies the equation*

$$u(x) = \lambda \int_0^1 K(x, \xi) u(\xi) d\xi, \tag{6}$$

where $K(x, \xi)$ is given by the formula (5), *then u has continuous first and second derivatives and satisfies the boundary problem*

$$\frac{d^2u}{dx^2} + \lambda u = 0,\ u(0) = 0,\ u(1) = 0.$$

Proof.—x and ξ range from 0 to 1 and $K(x, \xi)$ is discontinuous at $x = \xi$. Let us then write (6) in the form

$$u(x) = \lambda \int_0^x K_1(x, \xi) u(\xi) d\xi + \lambda \int_x^1 K_0(x, \xi) u(\xi) d\xi.$$

Now we may apply the general rule for the differentiation of a definite integral with respect to a parameter.[1] Then

$$\frac{du}{dx} = \lambda \int_0^x K_1'(x, \xi) u(\xi) d\xi + \lambda \int_x^1 K_0'(x, \xi) u(\xi) d\xi,$$

since

$$K_0(x, x) = K_1(x, x).$$

Moreover, since $K_1'(x, \xi)$ and $K_0'(x, \xi)$ are continuous in their respective intervals, we see that $\dfrac{du}{dx}$ is continuous. A second differentiation gives

$$\frac{d^2u}{dx^2} = \lambda \int_0^x K_1''(x, \xi) u(\xi) d\xi + \lambda \int_x^1 K_0''(x, \xi) u(\xi) d\xi$$
$$+ \lambda K_1'(x, x) u(x) - \lambda K_0'(x, x) u(x) = -\lambda u(x),$$

since by our hypothesis on K we have $K_1'' = K_0'' = 0$ and $K_1'(x, x) - K_0'(x, x) = -1$. Moreover, since $u(x)$ is continuous, we have u'' continuous. We have further

$$u(0) = \lambda \int_0^1 K(0, \xi) u(\xi) d\xi = 0,$$

[1] See Goursat-Hedrick, "Mathematical Analysis," vol. 1, §97.

since $K(0, \xi) = 0$, and

$$u(1) = \lambda \int_0^1 K(1, \xi)u(\xi)d\xi = 0,$$

since $K(1, \xi) = 0$.

We know the solution of the given boundary problem (4). This knowledge combined with Theorems IV and V gives the following:

Theorem VI.—*Only when* $\lambda = n^2\pi^2$(*n, an integer*) *does the integral equation*

$$u(x) = \lambda \int_0^1 K(x, \xi)u(\xi)d\xi$$

have a solution not identically zero:

$$u(x) = B \sin n\pi x.$$

If we compare these results with the results obtained in the preceding chapter for the general homogeneous integral equation, we see that the characteristic constants for this particular problem are $\lambda = n^2\pi^2 \equiv \lambda_n$ and that they are of index $q = 1$. The fundamental function belonging to λ_n is

$$\varphi(x) = \sin n\pi x.$$

The kernel is a symmetric one, so that

$$K(x, \xi) = K(\xi, x),$$

and therefore the associated equation is identical with the original one and hence has the same solutions. The associated fundamental functions are, therefore,

$$\overline{\varphi}(x) = \sin n\pi x.$$

II. Constrained Vibrations of an Elastic String

31. The Differential Equations of the Problem.—Let us suppose that an exterior force $\mu H(x, t)$ acts on each particle of mass μ in the y direction. Then it is known from the mathematical theory[1] of vibrating strings that the equations of motion of the string are

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + H(x, t), \tag{7}$$

(8) $y(0, t) = 0,$ $y(1, t) = 0,$

(9) $y(x, 0) = g(x),$ $y_t(x, 0) = 0.$

Let us suppose now that $H(x, t)$ is harmonic, that is,

$$H(x, t) = C^2 r(x) \cos (\beta t + \gamma) \quad (c \neq 0).$$

Let us find, if possible, a solution of the form

$$y(x, t) = u(x) \cos (\beta t + \gamma).$$

Substitute this value of y in (7) and put $c^2\lambda = \beta^2$. We find

$$\frac{d^2 u}{dx^2} + \lambda u + r(x) = 0, \tag{10}$$

while from (8) we derive the boundary conditions

$$u(0) = 0, \quad u(1) = 0. \tag{11}$$

32. Equivalence Between the Boundary Problem and a Linear Integral Equation.—Construct as before the Green's function $K(x, \xi)$. Then

$$\frac{d^2 K}{dx^2} = 0, \qquad \frac{d^2 u}{dx^2} = -\lambda u - r.$$

Multiply the first of these by u, the second by $-K$ and add. We obtain

$$uK'' - Ku'' = \lambda u K + rK,$$

which may be written

$$\frac{d}{dx}(uK' - Ku') = \lambda u K + rK.$$

Proceed as before with the integration from 0 to ξ and from ξ to 1. We find

$$u(\xi) = \lambda \int_0^1 K(x, \xi) u(x) dx + \int_0^1 K(x, \xi) r(x) dx.$$

[1] Weber, *Loc. cit.*, §83.

Interchange x and ξ. Then, on account of the symmetry of K, we have

$$u(x) = \lambda \int_0^1 K(x, \xi)u(\xi)d\xi + \int_0^1 K(x, \xi)r(\xi)d\xi.$$

If, now, we put

$$\int_0^1 K(x, \xi)r(\xi)d\xi = f(x) \qquad (12)$$

we have

$$u(x) = f(x) + \lambda \int_0^1 K(x, \xi)u(\xi)d\xi, \qquad (13)$$

which is a non-homogeneous linear integral equation of the second kind. It is satisfied by every solution of the boundary problem given by (10) and (11).

If we will proceed exactly as in the case of the homogeneous equation, we can now show conversely that if u is continuous and satisfies the equation (13), where $f(x)$ is given by (12), then u has continuous first and second derivatives and satisfies the differential equation (10) and the boundary conditions (11).

33. Remarks on Solution of the Boundary Problem.—Equations (6) and (13) have the same kernel, namely, the Green's function which we have constructed. Knowing that the characteristic constants for (6), and hence for (13), are $\lambda_n = n^2\pi^2$, we obtain from the general theory the following results for (13):

Case I.—If $\lambda \neq n^2\pi^2$, (13) has a unique solution.

Case II.—If $\lambda = n^2\pi^2$, there is, in general, no solution. A solution exists only when the condition

$$\int_0^1 f(x) \sin n\pi x dx = 0 \qquad (14)$$

is satisfied. This condition is what

$$\int_a^b f(x)\overline{\varphi}_\alpha(x)dx = 0$$

becomes for this special problem. If (14) is satisfied, then (13) has ∞^1 solutions.

The boundary problem

$$\frac{d^2u}{dx^2} + \lambda u + r(x) = 0, \; u(0) = 0, \; u(1) = 0$$

was shown to be equivalent to the non-homogeneous integral equation (13) if

(15) $$f(x) = \int_0^1 K(x, \xi) r(\xi) d\xi.$$

This enables us, from Cases I and II for the integral equation, to state that, when $\lambda \neq n^2\pi^2$, the boundary problem has a unique solution, and when $\lambda = n^2\pi^2$, there is, in general, no solution, but that when (14) is satisfied, there are ∞^1 solutions. If, now, in (14) we substitute for $f(x)$ its value as given by (15), this condition becomes

$$\int_0^1 \int_0^1 K(x, \xi) r(\xi) \sin n\pi x d\xi dx = 0.$$

Interchange the order of integration and remember that $\sin n\pi x$ is a solution of the homogeneous equation for $\lambda = \lambda_n$:

$$\sin n\pi x = \lambda_n \int_0^1 K(x, \xi) \sin n\pi\xi d\xi$$

or

$$\sin n\pi\xi = \lambda_n \int_0^1 K(x, \xi) \sin n\pi x dx$$

[since $K(x, \xi) = K(\xi, x)$], then the double integral becomes

$$\frac{1}{\lambda_n} \int r(\xi) \sin n\pi\xi d\xi = 0.$$

That is, in order that the boundary problem (10) may have a solution, it is necessary and sufficient that $r(x)$ satisfies the equation

$$\int_0^1 r(x) \sin n\pi x dx = 0.$$

III. Auxiliary Theorems on Harmonic Functions

34. Harmonic Functions.—For the solution later of the two boundary problems of the potential theory known as *Dirichlet's problem* and *Neumann's problem* we shall need certain auxiliary theorems on *harmonic functions*, that is, functions of class C'' satisfying $\Delta u = 0$:

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

35. Definitions about Curves.—A curve C

$$C: \qquad x = \varphi(t), \quad y = \psi(t), \quad t_o \leq t \leq t_1$$

is said to be *continuous* if φ and ψ are continuous on $[t_o, t_1]$. We write this

$$C^c \cdot\sim\cdot \varphi^c \cdot \psi^c.$$

The symbol $\cdot\sim\cdot$ is read *is equivalent to* or *implies and is implied by*. The symbol $\cdot$ is read *and*. Further, we say that the curve C is of class c' if φ and ψ have continuous first derivatives and these first derivatives φ' and ψ' do not vanish simultaneously on $[t_o, t_1]$. We write this

$$C^{c'} \cdot\sim\cdot \varphi^{c'} \cdot \psi^{c'} \cdot (\varphi', \psi') \neq (0, 0).$$

Such a curve is sometimes called a *smooth curve*. A curve of class c' has a definite positive tangent at every point. The condition $(\varphi', \psi') \neq (0, 0)$ excludes singular points. Every arc has a definite finite length. Thus we are assured that for smooth curves we can choose as parameter t the length s of arc and write

$$C: \qquad x = \xi(s), \quad y = \eta(s), \quad 0 \leq s \leq l$$
$$\xi'^2 + \eta'^2 = 1,$$

when l is the total length of arc.

In like manner, we define a curve of class C'':

$$C^{c''} \cdot \sim \cdot \varphi^{c''} \cdot \psi^{c''} \cdot (\varphi', \psi') \neq (0, 0).$$

Curves of class c'' have at every point a definite curvature,[1] which varies continuously from point to point.

36. Green's Theorem.—Let C be a curve with the following properties:

1) It is closed.
2) It has no multiple points.
3) It is of class c''. Hence we can represent it with the arc s as parameter:

$$x = \xi(s), \quad y = \eta(s), \quad 0 \leq s \leq l.$$

4) There exists a positive integer m such that every line parallel to the y or x axis meets the curve in at most m points.

According to Jordan's theorem[2] such a curve divides the plane into an interior and an exterior region. Let us denote by

I, the interior plus the boundary C,
by E, the exterior plus the boundary C.

Then Green's theorem[3] may be stated as follows:

Green's Theorem.—*If $P(x, y)$ and $Q(x, y)$ are of class c' on I, then*

$$\int_{(C)} \left[P(x, y)dx + Q(x, y)dy \right] = \int_I \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \tag{16}$$

where the line integral is taken in the positive sense around C.

[1] See Goursat-Hedrick, "Mathematical Analysis," vol. 1, §205, note.

[2] See Osgood, "Funktionentheorie," 2nd ed., p. 171.

[3] See Goursat-Hedrick, "Mathematical Analysis," vol. 1, §126, for a proof of Green's theorem.

If we put $P = v\dfrac{\partial u}{\partial y}$, $Q = -v\dfrac{\partial u}{\partial x}$, then Green's theorem becomes: If v is of class c' and u of class c'' on I, then

$$(17)\quad \int_{(C)} v\left(\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy\right) = -\iint_I v\Delta u\,dx\,dy - \iint_I \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dx\,dy.$$

The left-hand side can be simplified if we introduce the idea of the *directional derivative* of a function $f(x, y)$.

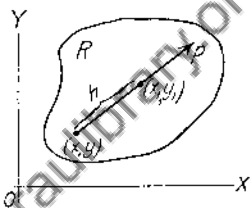


FIG. 7.

Let $f(x, y)$ be defined at every point of a region R of the xy-plane. The directional derivative is defined[1] as follows (see figure):

$$\frac{\partial f}{\partial p} = \lim_{h\to 0}\frac{f(x_1, y_1) - f(x, y)}{h},$$

if such limit exists.

If $f(x, y)$ is of class c', then[2]

$$f(x_1, y_1) - f(x, y) = f_x(x, y)(x_1 - x) + f_y(x, y)(y_1 - y) + \alpha(x_1 - x) + \beta(y_1 - y),$$

where α and β approach 0 with $x_1 - x$ and $y_1 - y$.

Divide both members of this last equality by h and then let h approach zero. We obtain

$$(18)\qquad \frac{\partial f(x, y)}{\partial p} = \frac{\partial f}{\partial x}\cos(px) + \frac{\partial f}{\partial y}\cos(py).$$

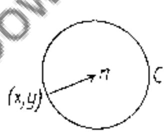


FIG. 8.

Let now n denote the direction of the inner normal to our curve C at a point (x, y) of C. Then, according to (18),

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos(nx) + \frac{\partial u}{\partial y}\cos(ny).$$

[1] See OSGOOD, "Differential and Integral Calculus," rev. ed., p. 308, 1910.

[2] See OSGOOD, *Loc. cit.*, p. 292.

But

$$\cos(nx) = -\frac{dy}{ds}, \cos(ny) = \frac{dx}{ds},$$

then

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{\partial y}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds}.$$

Thus, in Green's theorem, we may put

$$\int_{(C)} v\left(\frac{\partial u}{\partial y}\frac{dx}{ds} - \frac{\partial u}{\partial x}\frac{dy}{ds}\right)ds = \int_{(C)} v\frac{\partial u}{\partial n}ds.$$

We have then the following theorem:

Theorem VII.—*If u belongs to the class c'' and v to the class c' on I, then*

(19) $$\int_{(C)} v\frac{\partial u}{\partial n}ds = \int\int_I v\Delta u dxdy - \int\int_I\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dxdy.$$

We now apply this theorem to two special cases:

Case I.—$v = 1$, $\Delta u = 0$ on I. Then we have the theorem:

Theorem VIII.—*If u is harmonic on I, then*

(20) $$\int_{(C)} \frac{\partial u}{\partial n}ds = 0.$$

Case II.—$v = u$, $\Delta u = 0$ on I. Then we have the

Theorem IX.—*If u is harmonic on I, then*

$$-\int_{(C)} u\frac{\partial u}{\partial n}ds = \int\int_I\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\}dxdy.$$

Corollary I.—*If u is harmonic on I and vanishes along the boundary of C, then $u \equiv 0$ on I.*

For, by these hypotheses,

$$= 0 \int\int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dxdy,$$

hence $\frac{\partial u}{\partial x} \equiv 0$, $\frac{\partial u}{\partial y} \equiv 0$ and, therefore, u = constant on I. But u is continuous and vanishes on the boundary C of I, hence $u \equiv 0$.

Corollary II.—If u is harmonic on I and $\frac{\partial u}{\partial n} = 0$ on the boundary C of I, then u = constant on I.

Notice that in this case we cannot draw the conclusion $u \equiv 0$.

37. The Analogue of Theorem IX for the Exterior Region. Under the assumption that u is harmonic on E, it follows that u is harmonic in a region E_o exterior to C and interior to a circle S around the origin which includes C. From Theorem IX we would then have

$$(21) \quad \int\int_{E_o} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy = -\int_C u\frac{\partial u}{\partial n_e} ds - \int_S u\frac{\partial u}{\partial n} ds,$$

where the normals must be drawn toward the interior of the region E_o. We now let the radius r of the circle S approach ∞ and examine the limits of the three terms in (21). The first of the single integrals remains unchanged. In order to see what happens to the second, make a transformation to polar coordinates. Then $u(x, y)$ becomes $U(r, \theta)$, $ds = rd\theta$, and $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial r}$, since the normal is opposite in direction to the radius r. Then the second single integral becomes

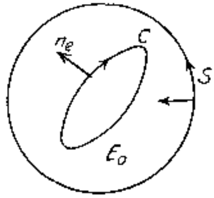


FIG. 9.

$$-\int_0^{2\pi} rU\frac{\partial U}{\partial r}d\theta.$$

Let us now add the assumption that

$$\lim_{r\to\infty} rU\frac{\partial U}{\partial r} = 0, \text{ uniformly with respect to } \theta.$$

Then $-\int_0^{2\pi} rU\frac{\partial U}{\partial r}d\theta$ approaches 0 as r approaches ∞, and, accordingly, also the double integral on the left-hand side of (21) approaches a determinate limit denoted by $\iint\limits_E \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]dxdy$. Hence we have the theorem:

Theorem X.—*If u is harmonic on E and*

(22) $\lim\limits_{r\to\infty} rU\frac{\partial U}{\partial r} = 0$, *uniformly with respect to θ, then*

(23) $$\iint\limits_E \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]dxdy = -\int\limits_{\overline{C}} u\frac{\partial u}{\partial n_e}ds.$$

As before, we obtain the two corollaries:

Corollary I.—If u is harmonic on E and (22) *still holds, and, moreover, $u = 0$ on C, then $u \equiv 0$ on E.*

Corollary II.—If u is harmonic on E and (22) *still holds, and, moreover, $\frac{\partial u}{\partial n_e} = 0$ on C, then u = constant on E.*

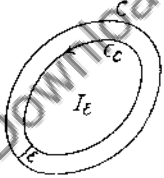


FIG. 10.

38. Generalization of the Preceding.—In the sequel we shall need a modification of the preceding theorems for the case where u is harmonic on the interior but not upon the boundary of C. Let us designate the interior without the boundary C by I'. Construct a closed curve C_ϵ.

$$C_\epsilon: \qquad x = \xi(s, \epsilon), \quad y = \eta(s, \epsilon)$$

of the same character as C interior to C and such that

$$\lim_{\epsilon\to 0} C_\epsilon = C, \text{ uniformly as to } s.$$

Denote by I_ϵ the interior of C_ϵ plus the boundary C_ϵ. Theorem IX applies for this region and we have

$$\iint\limits_{I_\epsilon}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dxdy=-\int\limits_{\overline{C}_\epsilon} u\frac{\partial u}{\partial n}\,ds.$$

Let us now impose the following three conditions (A):

1) $\lim\limits_{\epsilon\to 0} u = u_i$, uniformly as to s.

2) $\lim\limits_{\epsilon\to 0}\dfrac{\partial u}{\partial n}=\dfrac{\partial u_i}{\partial n}$, uniformly as to s.

3) $\left|\dfrac{\partial u}{\partial x}\right|,\left|\dfrac{\partial u}{\partial y}\right|$ are bounded on I'.

Then we obtain the following theorem:

Theorem XI.—*If u is harmonic on I' and satisfies the three conditions (A), then*

$$(24)\qquad \iint\limits_{I}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dxdy=-\int\limits_{\overline{C}} u_i\frac{\partial u_i}{\partial n}\,ds.$$

The following two corollaries follow as before.

Corollary I.—If we add to the hypotheses of the theorem that $u_i = 0$ along C, then $u \equiv 0$ on I'.

Corollary II.—If we add to the hypotheses of the theorem that $\dfrac{\partial u_i}{\partial n} = 0$ along C, then u = constant on I'.

Make hypotheses similar to (A) for the exterior and call them (B). Then we get for the exterior minus the boundary C, which we designate by E' a corresponding theorem with two corollaries.

Theorem XII.—*If u is harmonic on E' and satisfies the conditions (B), then*

$$\iint\limits_{E}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dxdy=-\int\limits_{\overline{C}} u_e\frac{\partial u_e}{\partial n_e}ds.$$

Corollary I.—If we add to the hypotheses of the theorem that $u_e = 0$ *along* C, *then* $u \equiv 0$ *on* E'.

Corollary II.—If we add to the hypotheses of the theorem that $\dfrac{\partial u_e}{\partial n_e} = 0$ on C, *then* u = *constant on* E'.

IV. Logarithmic Potential of a Double Layer

39. Definition.—We suppose that C

$$C: \qquad x = \xi(s), \quad y = \eta(s), \quad s = 0, \; . \; . \; . \; , l$$

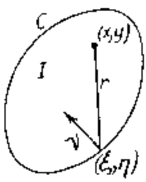


Fig. 11.

has the properties 1) . . . 4) of §36. Let (ξ, η) be a point on C, (x, y) a fixed point not on C, r their distance:

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2},$$

ν the inner normal to C at (ξ, η), $\mu(s)$ a continuous function on C. Then the definite integral

$$(25) \qquad w(x, y) = \int_0^l \mu(s) \frac{\partial \log \frac{1}{r}}{\partial \nu} ds$$

for physical reasons is called the *logarithmic potential of a double layer* of density μ (s) distributed over the curve C

We obtain as follows a more explicit form for $\dfrac{\partial}{\partial \nu} \log \dfrac{1}{r}$.

$$(26) \qquad \frac{\partial}{\partial \nu} \log \frac{1}{r} = \cos(\nu\xi) \frac{\partial}{\partial \xi} \log \frac{1}{r} + \cos(\nu\eta) \frac{\partial}{\partial \eta} \log \frac{1}{r}$$

$$(27) \qquad = \frac{1}{r}\left[\frac{x - \xi}{r} \cos(\nu\xi) + \frac{y - \eta}{r} \cos(\nu\eta)\right]$$

$$= \frac{1}{r}\left[\cos(r\xi) \cos(\nu\xi) + \cos(r\eta) \cos(\nu\eta)\right]$$

$$= \frac{\cos(r\nu)}{r}.$$

But from (27), since $\cos(\nu\xi) = -\eta'(s)$, $\cos(\nu\eta) = \xi'(s)$, we have

$$\frac{\partial}{\partial\nu}\log\frac{1}{r} = \frac{\left[y - \eta(s)\right]\xi'(s) - \left[x - \xi(s)\right]\eta'(s)}{r^2}$$

Thus $w(x, y)$ may be written in the form

$$w(x, y) = \int_0^l \mu(s)\,\frac{\cos(r\nu)}{r}\,ds,$$

or

$$(28)\quad w(x, y) = \int_0^l \mu(s)\frac{\left[y - \eta(s)\right]\xi'(s) - \left[x - \xi(s)\right]\eta'(s)}{\left[x - \xi(s)\right]^2 + \left[y - \eta(s)\right]^2}\,ds,$$

which shows explicitly the dependence of the integrand upon the parameters x and y.

40. Properties of w(x, y) at Points Not on C.—Put

$$u(x, y) = \log\frac{1}{r}.$$

Then we have

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{r^2} + \frac{2(x - \xi)^2}{r^4}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{r^2} + \frac{2(y - \eta)^2}{r^4}$$

Whence by adding we find $\Delta u = 0$. That is, $\log\frac{1}{r}$ is harmonic. Further

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial\xi}\log\frac{1}{r}\right) = \frac{\partial}{\partial\xi}\left(\frac{\partial^2}{\partial x^2}\log\frac{1}{r}\right)$$

and

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial\xi}\log\frac{1}{r}\right) = \frac{\partial}{\partial\xi}\left(\frac{\partial^2}{\partial y^2}\log\frac{1}{r}\right).$$

Whence, adding, we find

$$\Delta\left(\frac{\partial}{\partial \xi} \log \frac{1}{r}\right) = \frac{\partial}{\partial \xi}\left(\Delta \log \frac{1}{r}\right) = \frac{\partial}{\partial \xi} \Delta u = 0.$$

Therefore $\frac{\partial}{\partial \xi} \log \frac{1}{r}$ is harmonic. Similarly, we can show that $\frac{\partial}{\partial \eta} \log \frac{1}{r}$ is harmonic. Hence it follows from (26) that $\frac{\partial}{\partial \nu} \log \frac{1}{r}$ is harmonic. That is,

$$\Delta\left(\frac{\partial}{\partial \nu} \log \frac{1}{r}\right) = 0. \tag{29}$$

Let us now compute $\Delta w(x, y)$. We have

$$\begin{aligned} \frac{\partial^2}{\partial x^2} w(x, y) &= \frac{\partial^2}{\partial x^2} \int_0^l \mu(s) \frac{\partial}{\partial \nu} \log \frac{1}{r}\, ds \\ &= \int_0^l \mu(s) \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial \nu} \log \frac{1}{r}\right) ds, \end{aligned} \tag{30}$$

since the rules for the differentiation of a definite integral with respect to a parameter are applicable. Likewise, we have

$$\frac{\partial^2}{\partial y^2} w(x, y) = \int_0^l \mu(s) \frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial \nu} \log \frac{1}{r}\right) ds. \tag{31}$$

Add (30) and (31) and take account of (29). We find

$$\Delta w(x, y) = \int_0^l \mu(s) \Delta\left(\frac{\partial}{\partial \nu} \log \frac{1}{r}\right) ds = 0.$$

From the explicit expression (28) we see that $w(x, y)$ is single-valued and continuous with all of its derivatives at every point (x, y) not on C. Thus we have proved the following theorem:

Theorem XIII.—*The function*

$$w(x, y) = \int_0^l \mu(s) \frac{\partial}{\partial \nu} \log \frac{1}{r}\, ds$$

is single-valued and continuous with all of its derivatives at every point (x, y) *not on* C*, and in the same domain* w *is harmonic:*

$$\Delta w(x, y) = 0.$$

41. Behavior of w(x, y) on C.—In the preceding discussion (x, y) was supposed *not* to lie on C. Let now (x, y) coincide with a point (x_o, y_o) on C, corresponding to a value $s = s_o$, so that

$$x_o = \xi(s_o), \quad y_o = \eta(s_o).$$

Then the integrand becomes indeterminate. The indeterminate expression which we desire to investigate is

$$\frac{\cos(r_o\nu)}{r_o} = \frac{\left[\eta(s_o) - \eta(s)\right]\xi'(s) - \left[\xi(s_o) - \xi(s)\right]\eta'(s)}{\left[\xi(s_o) - \xi(s)\right]^2 + \left[\eta(s_o) - \eta(s)\right]^2}.$$

To evaluate, apply Taylor's remainder theorem to both numerator and denominator, stopping at the derivatives of the second order. A factor $(s - s_o)^2$ appears in both and cancels. The limit of what remains as $s \to s_o$ is

$$\frac{\xi'\eta'' - \xi''\eta'}{2(\xi'^2 + \eta'^2)} = \frac{1}{2R_o},$$

where R_o is the radius of curvature of C at (x_o, y_o). $w(x, y)$ is thus defined and has a determinate finite value at every point (x_o, y_o) of C:

$$w(x_o, y_o) = \int_0^l \mu(s) \frac{\left[\eta(s_o) - \eta(s)\right]\xi'(s) - \left[\xi(s_o) - \xi(s)\right]\eta'(s)}{\left[\xi(s_o) - \xi(s)\right]^2 + \left[\eta(s_o) - \eta(s)\right]^2} ds.$$

The existence of $w(x, y)$ on the boundary of C does not, however, imply that $w(x, y)$ remains continuous, as (x, y) crosses the boundary C. On the contrary, $w(x, y)$ undergoes a discontinuity as (x, y) crosses C. Let us denote by P_i an interior point of C and by P_e an exterior point (x, y) in the vicinity of $P_o(x_o, y_o)$. If P_i approaches P_o, then $w(x, y)$ approaches a definite finite limit $w_i(x_o, y_o)$. If P_e approaches P_o, then $w(x, y)$ approaches a definite finite limit $w_e(x_o, y_o)$. Between these quantities and $w(x_o, y_o)$ defined above the following relations hold:

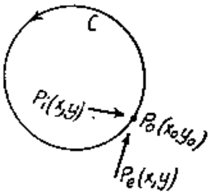


FIG. 12.

$$
(32) \qquad \begin{aligned} w_i(x_o, y_o) &= w(x_o, y_o) + \pi.\mu(s_o) \\ w_e(x_o, y_o) &= w(x_o, y_o) - \pi.\mu(s_o).^1 \end{aligned}
$$

For future use we give here a proof for the special case $\mu(s) \equiv 1$. Let us use the notation

$$\zeta = \xi + i\eta, \quad z = x + iy$$

and consider the integral H along C:

$$H_c = \int_c \frac{d\zeta}{\zeta - z}.$$

By Cauchy's first integral theorem

$$H_c = \begin{cases} 2\pi i, & xy \text{ an interior point of } C. \\ 0, & xy \text{ an exterior point of } C. \end{cases}$$

If we separate the real and the imaginary parts of the integral, we obtain

$$H_c = \int \frac{(\xi - x)d\xi + (\eta - y)d\eta}{(x - \xi)^2 + (y - \eta)^2} + i \int \frac{(y - \eta)d\xi - (x - \xi)d\eta}{(x - \xi)^2 + (y - \eta)^2} \equiv V_c + iW_c.$$

[1] For a proof of these statements see HORN, "Einfuhrung in die Theorie der Partiellen Differential-Gleichungen," §52.

Hence, by equation (28), $w(x, y)$ is the coefficient of i in the integral H_c. Thus we get

$$w(x, y) = \begin{cases} 2\pi, & xy \text{ an interior point of } C. \\ 0, & xy \text{ an exterior point of } C. \end{cases}$$

But if $w(x, y) = 2\pi$ constantly as x, y varies, then

$$\lim_{P_i \to P_o} w(x, y) = w_i(x_o, y_o) = 2\pi.$$

In like manner

$$\lim_{P_e \to P_o} w(x, y) = w_e(x_o, y_o) = 0.$$

We have now, in order to complete the proof, to compute $w(x_o, y_o)$ for a point (x_o, y_o) *on* C. Draw about P_o an arc α of a circle cutting C in Q and R. This arc will subtend an angle ω at P_o. Designate by C' the path C minus the arc QP_oR. Then $H_{C'} + H_\alpha = 0$, since P_o is an exterior point for this path. But in the elements of the theory of functions of a complex variable it is shown that

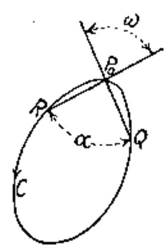


FIG. 13.

$$H_\alpha = \int_\alpha \frac{d\zeta}{\zeta - z_o} = -\varpi i,$$

whence

$$H_{C'} = i\varpi.$$

Therefore,

$$W_{C'} = \varpi.$$

Let the radius of the arc α be ρ. Then, as $\rho \to 0$,

$$W_{C'} \to W_C = w(x_o, y_o).$$

Now the integral W_C is convergent as shown above and $\varpi \to \pi$. Therefore

$$w(x_o, y_o) = \pi.$$

The equations (32) are satisfied for these values of $w(x_o, y_o)$, $w_i(x_o, y_o)$, and $w_e(x_o, y_o)$ for the special case $\mu(s) \equiv 1$.

42. Behavior of $\frac{\partial w}{\partial n}$ on the Boundary C and at Infinity.—At the point $P_o(x_o, y_o)$ draw the interior normal n_i and take on it a point $P_i(x, y)$. From Theorem XIII we know that at P_i, $\frac{\partial w(x, y)}{\partial n_i}$ is definite and finite. Let the point P_i approach P_o along n_i and put

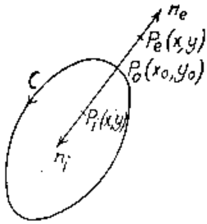


FIG. 14.

$$\lim_{P_i \to P_o} \frac{\partial w}{\partial n_i} = \frac{\partial w_i}{\partial n_i}, \text{ if the limit exists,}$$

$$\lim_{P_e \to P_o} \frac{\partial w}{\partial n_i} = \frac{\partial w_e}{\partial n_i}, \text{ if the limit exists,}$$

P_e designating a point on the prolongation of n_i beyond P_o. Then the following theorem is true.[1]

Theorem XIV.—*If one of the two limits* $\frac{\partial w_i}{\partial n_i}$, $\frac{\partial w_e}{\partial n_i}$, *exists, then also the other exists and*

$$\frac{\partial w_i}{\partial n_i} = \frac{\partial w_e}{\partial n_i}. \tag{33}$$

If we denote by n_e the exterior normal to C at P_o, then

$$\frac{\partial}{\partial n_i} = -\frac{\partial}{\partial n_e}$$

from the definition of the directional derivative. Hence (33) may also be written

$$\frac{\partial w_i}{\partial n_i} = -\frac{\partial w_e}{\partial n_e}. \tag{34}$$

If we introduce polar coordinates r, θ, then $w(x, y)$ becomes $W(r, \theta)$. Then the following theorem is true:[1]

[1] For proof, see HORN, *Loc. cit.*, §54.

Theorem XV.—

$$\lim_{r \to \infty} W(r, \theta) = 0, \textit{ uniformly as to } \theta.$$

$$\lim_{r \to \infty} rW\frac{\partial W}{\partial r} = 0, \textit{ uniformly as to } \theta.$$

43. Case Where w_i or w_e Vanish Along C.—We can now prove the following theorem:

Theorem XVI.—*If $w_i(x, y) \equiv 0$ along C, then*

$$\mu(s) \equiv 0.$$

Proof.—Theorem XIII assures us that $w(x, y)$ is harmonic on I'. By hypothesis $w_i(x, y) \equiv 0$ along C. Taking it for granted that the conditions (A) are always satisfied by w, it follows from Corollary I to Theorem XI that

$$w(x, y) \equiv 0 \text{ on } I'.$$

Hence it follows at once that $\frac{\partial w}{\partial n_i} \equiv 0$ for every interior point of C. Therefore, $\frac{\partial w_i}{\partial n_i}$ exists along C and vanishes there. Hence, also by Theorem XIV, $\frac{\partial w_e}{\partial n_e}$ exists along C and vanishes there. By Theorem XV, $\lim_{r \to \infty} rW\frac{\partial W}{\partial r} = 0$, uniformly as to θ. Hence, if we take it for granted that the conditions (B) on w are satisfied, then it follows from Corollary II to Theorem XII that w is constant on E'. We see from Theorem XV that this constant must be zero:

$$w(x, y) \equiv 0 \text{ on } E'.$$

Whence it follows that $w_e(x, y) = 0$ on C. If we now apply (32) we find

$$w_i(x_o, y_o) - w_e(x_o, y_o) = 2\pi\mu(s_o) = 0,$$

uniformly as to s_o. Therefore

$$\mu(s_o) \equiv 0, \text{ uniformly as to } s_o.$$

A similar theorem holds with respect to the limit w_e:

Theorem XVII.—*If* $w_e(x, y) \equiv 0$ *along* C, *then*

$$\mu(s) = constant.$$

Proof.—Theorem XIII assures us that w is harmonic on E'. By hypothesis $w_e(x, y) \equiv 0$ along C. Furthermore, $\lim_{r\to\infty} rW\frac{\partial W}{\partial r} = 0$, uniformly as to θ, according to Theorem XV. Taking it for granted that the conditions (B) of §38 are always satisfied by w, it follows from Corollary I to Theorem XII that $w(x, y) \equiv 0$ on E'. Hence it follows at once that $\frac{\partial w}{\partial n_e} \equiv 0$ for every point exterior to C. Therefore $\frac{\partial w_e}{\partial n_e}$ exists along C and vanishes there. Hence also, by Theorem XIV, $\frac{\partial w_i}{\partial n_i}$ exists along C and vanishes there. Taking it for granted that the conditions (A) are always satisfied by w, it follows from Corollary II to Theorem XI that

$$w(x, y) = \text{constant on } I'.$$

Whence it follows that $w_i(x, y) = \text{constant}$ on C. If we now apply (32), we find

$$w_i(x_o, y_o) - w_e(x_o, y_o) = 2\pi\mu(s_o) = \text{constant},$$

uniformly as to s_o. Therefore,

$$\mu(s_o) = \text{constant, uniformly as to } s_o.$$

V. Fredholm's Solution of Dirichlet's Problem

44. Dirichlet's Problem.—We formulate Dirichlet's problem as follows:

Given.—1) A closed curve C

$$C: \quad x = \xi(s),\ y = \eta(s),\ 0 \leq s \leq l$$

(35) $$\xi(0) = \xi(l),\ \eta(0) = \eta(l)$$

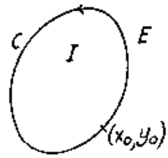


FIG. 15.

with the properties 1) . . . 4) of §36.

2) A function $F(s)$, continuous on C for $0 \leq s \leq l$, and

(36) $$F(0) = F(l)$$

Required.—A function $u(x, y)$ such that

α) u is harmonic on I', and

β) $u_i(x_o, y_o) = F(s_o)$, uniformly as to s_o,

where $u_i(x_o, y_o)$ is the limit approached by $u(x, y)$ as the point (x, y) approaches from the interior a point (x_o, y_o) of parameter s_o on the boundary C.

45. Reduction to an Integral Equation. *First Method.*—The function $w(x, y)$:

$$w(x, y) = \int_0^l \mu(s) \frac{\partial}{\partial \nu} \log \frac{1}{r}\, ds$$

is harmonic on I' and thus satisfies the condition (α) for every choice of $\mu(s)$ for which $\mu(s)$ is continuous on $[0l]$ and

$$\mu(0) = \mu(l).$$

The function $w(x, y)$ will then furnish a solution of Dirichlet's problem if $\mu(s)$ can be so determined that the condition (β) is satisfied:

(37) $$w_i(x_o, y_o) = F(s_o), \text{ uniformly as to } s_o.$$

From the first of equations (32) we have

$$w_i(x_o, y_o) = w(x_o, y_o) + \pi\mu(s_o).$$

The substitution of this value of w_i in (37) gives

(38) $$w(x_o, y_o) + \pi\mu(s_o) = F(s_o).$$

If in this equality we substitute for $w(x_o, y_o)$ its explicit expression as given in §41, we obtain

$$(39)\quad \int_0^l \mu(s)\frac{\Big[\eta(s_o) - \eta(s)\Big]\xi'(s) - \Big[\xi(s_o) - \xi(s)\Big]\eta'(s)}{\Big[\xi(s_o) - \xi(s)\Big]^2 + \Big[\eta(s_o) - \eta(s)\Big]^2}ds + \pi\mu(s_o) = F(s_o).$$

Divide through by π. Then (39) becomes an integral equation for the determination of $\mu(s_o)$ with the kernel

$$(40)\quad K(s_o, s) = \frac{1}{\pi}\frac{\Big[\eta(s_o) - \eta(s)\Big]\xi'(s) - \Big[\xi(s_o) - \xi(s)\Big]\eta'(s)}{\Big[\xi(s_o) - \xi(s)\Big]^2 + \Big[\eta(s_o) - \eta(s)\Big]^2}$$

Put $\dfrac{F(s_o)}{\pi} = f(s_o)$. Then the integral equation takes the standard form

$$(41)\qquad \mu(s_o) = f(s_o) - \int_0^l K(s_o, s)\mu(s)ds.$$

This is a special case of the integral equation with a parameter λ, for which $\lambda = -1$.

Thus, in order that $u = w(x, y)$ may be a solution of Dirichlet's problem, it is *necessary* that the density $\mu(s)$ satisfy the integral equation (41). This condition is also *sufficient*. For, suppose $\mu(s)$ to be a continuous solution of (41). Then we have

$K(0, s) = K(l, s)$ on account of (35), and
$f(0) = f(l)$ on account of (36), and hence
$\mu(0) = \mu(l)$ from (41).

The function $w(x, y)$ formed with this function $\mu(s)$ will then satisfy (38) and, therefore, also (37) and, consequently, $u = w(x, y)$ will be a solution of Dirichlet's problem.

Second Method. From equations (32) we find

$$(42) \qquad \begin{aligned} w_i(x_o, y_o) - w_e(x_o, y_o) &= 2\pi\mu(s_o), \text{ and} \\ w_i(x_o, y_o) + w_e(x_o, y_o) &= 2w(x_o, y_o), \end{aligned}$$

which by §41 $$= 2\int_0^l \mu(s)\frac{\cos(r_o\nu)}{r_o}ds.$$

We now seek as a solution of Dirichlet's problem a function u which is harmonic everywhere except on C, and which upon C satisfies the condition

$$(43) \qquad u_i + hu_e = F(s_o) + hG(s_o),$$

where h is an arbitrary parameter. For $h = 0$, we have the interior problem, and for $h = \infty$ the exterior problem.

The function $w(x, y)$ is harmonic everywhere except on C. This function will then be a solution of Dirichlet's problem, provided $\mu(s)$ can be so determined that $u = w(x, y)$ satisfies (43). Solve now the equations (42) for $u_i = w_i$ and $u_e = w_e$ and substitute in (43). We find

$$(44) \qquad \mu(s_o) = f(s_o) + \lambda\int_0^l \mu(s)\frac{\cos(r_o\nu)}{\pi r_o}ds$$

where $$f(s_o) = \frac{hG(s_o) + F(s_o)}{\pi(1 - h)}, \ \lambda = \frac{h+1}{h-1}.$$

This is an integral equation of the second kind with a parameter λ. The kernel is $\frac{\cos(r_o\nu)}{\pi r_o} \equiv K(s_o, s)$ [see (40)]. For $h = 0$, we have the interior problem, but for $h = 0$ we have $\lambda = -1$ and (44) reduces to (41). For $h = \infty$, we have the exterior problem, but for $h = \infty$ we have $\lambda = +1$, and (44) reduces to

$$(45) \qquad \mu(s_o) = \frac{G(s_o)}{\pi} + \int_0^l K(s_o, s)\mu(s)ds.$$

We remark that, if $F(s_o) \equiv 0$ on C and $h = 0$, then (44) becomes a homogeneous linear integral equation with $\lambda = -1$.

46. Solution of the Integral Equation. The kernel $K(s_o, s)$ is real, continuous, and $\neq 0$ in the region G,

$$G: \qquad 0 \leqq s_o \leqq l, \quad 0 \leqq s \leqq l,$$

and thus the preceding theory of integral equations is applicable. From Fredholm's first fundamental theorem we know that if $D(-1) \neq 0$ then (41) has one and only one continuous solution.

We show first that $D(-1) \neq 0$. For this purpose we use Fredholm's second fundamental theorem, from which it follows that the corresponding homogeneous integral equation

$$\mu(s_o) + \int_0^l K(s_o, s)\mu(s)ds = 0 \tag{46}$$

has no other solution than the trivial one $\mu \equiv 0$, if $D(-1) \neq 0$; while if $\lambda = -1$ is a root of $D(\lambda) = 0$ of index q, then (46) has ∞^q solutions. Hence, if (46) has no other solution than $\mu \equiv 0$, then $D(-1) \neq 0$.

Multiply the members of (46) by π. Then

$$\int_0^l \pi K(s_o, s)\mu(s)ds + \pi\mu(s_o) = 0,$$

uniformly as to s_o. This last equation, by (32), after taking account of our notations in §41 and equations (40), can be written

$$w_i(x_o, y_o) = 0, \text{ uniformly as to } s_o.$$

But, by Theorem XVI, if $w_i(x_o, y_o) = 0$ along C, then

$$\mu(s) \equiv 0.$$

Therefore, (46) has no other continuous solution than $\mu \equiv 0$ and, consequently,

$$D(-1) \neq 0.$$

Thus we see that $\lambda = -1$ is not a characteristic constant for the kernel $K(s_o, s)$. Hence (41) has one and only one continuous solution $\mu(s_o)$ representable by Fredholm's formula

$$\mu(s_o) = f(s_o) + \int_0^l \frac{D(s_o, s; -1)}{D(-1)} f(s)ds.$$

Therefore, $w(x, y)$ *with* $\mu = \mu(s_o)$ *is a solution of Dirichlet's problem.* We state this result in the following theorem:

Theorem XVIII.—*Given*

1) *A closed curve C*

$$C: \qquad x = \xi(s), \quad y = \eta(s), \quad 0 \leqq s \leqq l$$
$$\xi(0) = \xi(l), \quad \eta(0) = \eta(l)$$

with the properties 1) . . . 4) *of* §36;

2) *A function* $F(s)$, *continuous on* C *for* $0 \leqq s \leqq l$,

$$F(0) = F(l).$$

Then there exists a function $u(x, y)$ *such that*

α) u *is harmonic on* I', *and*

β) $u_i(x_o, y_o) = F(s_o)$, *uniformly as to* s_o, *where* $u_i(x_o, y_o)$ *is the limit approached by* $u(x, y)$ *as the point* (x, y) *approaches from the interior a point* (x_o, y_o) *of parameter* s_o *on the boundary* C. *This function is given by*

$$u = \int_0^l \mu(s)\frac{\partial}{\partial\nu} \log \frac{1}{r}\, ds,$$

where $\mu(s)$ *is the unique solution of* (41) *and is given by*

$$\mu(s_o) = f(s_o) + \int_0^l \frac{D(s_o, s; -1)}{D(-1)} f(s)ds,$$

where $\pi f(s_o) = F(s_o)$.

47. Index of $\lambda = 1$ for $K(s_o, s)$.—We have seen that $\lambda = -1$ is not a characteristic constant of $K(s_o, s)$. We shall now prove the following theorem:

Theorem XIX.—*$\lambda = +1$ is a characteristic constant of $K(s_o, s)$ of index 1.*

Proof.—According to Fredholm's second fundamental theorem, the statement to be proved is equivalent to the statement that the homogeneous equation

$$\mu(s_o) = \int_0^l K(s_o, s)\mu(s)ds \tag{47}$$

has ∞^1 solutions. Now, if we make use of equations (40) and (39) and the explicit expression for $w(x_o, y_o)$ given in §41, we see that (47) is equivalent to

$$w(x_o, y_o) = \pi\mu(s_o). \tag{48}$$

But from the proof given for the equations (32) for the special case $\mu(s) \equiv 1$, we see that $\mu(s) \equiv 1$ is a solution of (48), for when $\mu(s) \equiv 1$, we have

$$w(x_o, y_o) = \pi = \pi\mu(s_o).$$

This is a solution of (47) which is continuous and not identically zero. Hence $\lambda = 1$ is a *characteristic constant* of $K(s_o, s)$.

We will now determine the index of $\lambda = +1$ for $K(s_o, s)$. To this end we determine all solutions of (47) or of its equivalent (48). Now, on account of the second of equations (32), our equation (48) above reduces to

$$w_e(x_o, y_o) = 0.$$

But, according to Theorem XVII, the most general solution of this equation, and, therefore, of (47), is

$$\mu(s_o) = \text{constant}.$$

Therefore, (47) has ∞^1 solutions, which shows that the index of the characteristic constant $\lambda = +1$ of $K(s_o, s)$ is 1.

VI. Logarithmic Potential of a Simple Layer

48. Definition.—We have given a closed curve C

$$C: \qquad x = \xi(s), \quad y = \eta(s), \quad s = 0, \ldots, l$$

having the properties 1) . . . 4) of §36. Let r represent the distance between a point (ξ, η) on C and a point (x, y) not on C, $k(s)$ a continuous function on C. Then the definite integral

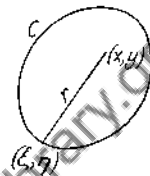


FIG. 16.

$$v(x, y) = \int_0^l k(s) \log \frac{1}{r}\, ds$$
$$= -\frac{1}{2}\int_0^l k(s) \log\left\{\left[x - \xi(s)\right]^2 + \left[y - \eta(s)\right]^2\right\} ds$$

is called the *logarithmic potential of a simple layer* of density $k(s)$ distributed over the curve C.

49. Properties of v(x, y).—

The $\log\left\{\left[x - \xi(s)\right]^2 + \left[y - \eta(s)\right]^2\right\}$ is a regular analytic function of x and y at all points not on C and hence the definite integral as a function of the parameters x and y defines a function $v(x, y)$ which is continuous with all of its derivatives in I' and E'. Further, $v(x, y)$ is harmonic in the same region. For, from the theory of definite integrals

$$\Delta v = \Delta \int_0^l k(s) \log \frac{1}{r} ds$$
$$= \int_0^l k(s) \Delta \log \frac{1}{r}\, ds.$$

But it has been previously shown that

$$\Delta \log \frac{1}{r} = 0.$$

Therefore

$$\Delta v(x, y) = 0.$$

Hence we have proved the following theorem:

Theorem XX.—*The logarithmic potential* $v(x, y)$:

$$v(x, y) = \int_0^l k(s) \log \frac{1}{r}\, ds \tag{49}$$

of a simple layer is continuous with its derivatives of all orders, and is harmonic, on I' *and* E':

$$\Delta v(x, y) = 0.$$

Let us now investigate the behavior of $v(x, y)$ on the boundary C at any point (x_o, y_o) with parameter s_o

$$x_o = \xi(s_o), \quad y_o = \eta(s_o).$$

From the explicit expression for $v(x, y)$:

$$v(x_o, y_o) = -\frac{1}{2}\int_0^l k(s) \times \log \left\{\left[\xi(s_o) - \xi(s)\right]^2 + \left[\eta(s_o) - \eta(s)\right]^2\right\} ds$$

we see that when $s = s_o$, the logarithm which appears in the integrand becomes infinite. Thus we see that we have to do with an improper definite integral. Apply Taylor's remainder theorem to the expression of which the logarithm is taken. We obtain

$$(s - s_o)^2 A,$$

where A does not become zero as s approaches s_o. Then the integrand becomes infinite like $\log (s - s_o)$ as $s \to s_o$. From the theory of improper definite integrals we know that then this definite integral remains definite and finite.

We state without proof[1] the two following theorems, in which $P_i(x, y)$ denotes an interior, $P_e(x, y)$ an exterior point in the vicinity of P_o.

[1] For a proof, see Horn, §53.

Theorem XXI.—*The limit of $v(x, y)$ as $P_i \to P_o$ exists:*

$$\lim_{P_i \to P_o} v(x, y) = v_i(x_o, y_o).$$

The limit of $v(x, y)$ as $P_e \to P_o$ exists:

$$\lim_{P_e \to P_o} v(x, y) = v_e(x, y);$$

and $$v_i(x_o, y_o) = v_e(x_o, y_o) = v(x_o, y_o).$$

That is, $v(x, y)$ remains continuous as the point $P(x, y)$ crosses the boundary C. In this respect the behavior of the function $v(x, y)$ is simpler than that of $w(x, y)$ for a double layer.

Construct for the point $P_o(x_o, y_o)$ the interior normal n_i to C and take on it the interior point P_i and on its exterior prolongation the exterior point P_e; then it follows from Theorem XX that $\dfrac{\partial v}{\partial n_i}$ exists at P_i and at P_e. For the approach of P_i and P_e toward P_o along the normal the following theorem holds:

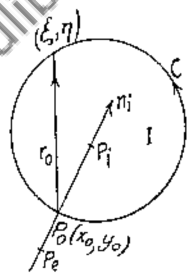


FIG. 17.

Theorem XXII.—*The limit of $\dfrac{\partial v}{\partial n_i}$ as $P_i \to P_o$ exists:*

$$\lim_{P_i \to P_o} \frac{\partial v}{\partial n_i} = \frac{\partial v_i}{\partial n_i} = -\pi k(s_o) + \int_0^l k(s) \frac{\cos (r_o n_i)}{r_o} ds.$$

The limit of $\dfrac{\partial v}{\partial n_i}$ as $P_e \to P_o$ exists:

$$\lim_{P_e \to P_o} \frac{\partial v}{\partial n_i} = \frac{\partial v_e}{\partial n_i} = +\pi k(s_o) + \int_0^l k(s) \frac{\cos (r_o n_i)}{r_o} ds,$$

where r_o is the distance from $P_o(x_o, y_o)$ to a variable point $\left[\xi(s), \eta(s)\right]$ of C.

By referring to the figure, we obtain for $\cos(r_o n_i)$ the expression

$$\cos(r_o n_i) = \cos(r_o x)\cos(n_i x) + \cos(r_o y)\cos(n_i y)$$
$$= -\frac{\xi(s) - x_o}{r_o}\eta'(s_o) + \frac{\eta(s) - y_o}{r_o}\xi'(s_o),$$

and hence

$$\frac{\cos(r_o n_i)}{r_o} = \frac{\Big[\eta(s) - \eta(s_o)\Big]\xi'(s_o) - \Big[\xi(s) - \xi(s_o)\Big]\eta'(s_o)}{\Big[\xi(s) - \xi(s_o)\Big]^2 + \Big[\eta(s) - \eta(s_o)\Big]^2}$$

VII. Fredholm's Solution of Neumann's Problem

50. Neumann's Problem.—We formulate the second boundary problem of the potential theory, or Neumann's problem, as follows:

Given.—

1) A closed curve C,

C: $$x = \xi(s), \quad y = \eta(s), \quad 0 \leqq s \leqq l$$
$$\xi(0) = \xi(l), \quad \eta(0) = \eta(l)$$

with the properties 1) . . . 4) of §36.

2) A function $F(s)$, continuous on C for $0 \leqq s \leqq l$:

$$F(0) = F(l).$$

Required.—A function $u(x, y)$ such that

α) $\Delta u = 0$ on I'.

β) $\dfrac{\partial u_i}{\partial n_i} = F(s_o)$, uniformly as to s_o.

51. Reduction to an Integral Equation. *First Method.*—The function $v(x, y)$ given by (49) is a function satisfying condition (α) for any choice of $k(s)$ which is continuous on

C. If it is possible to determine the function $k(s)$ so as to satisfy the condition (β):

$$(50) \qquad \frac{\partial v_i}{\partial n_i} = F(s_o), \text{ uniformly as to } s_o,$$

then we will have a solution of our problem.

Equation (50), written explicitly, according to Theorem XXII becomes

$$(51) \quad \int_0^l k(s) \frac{\left[\eta(s) - \eta(s_o)\right]\xi'(s_o) - \left[\xi(s) - \xi(s_o)\right]\eta'(s_o)}{\left[\xi(s) - \xi(s_o)\right]^2 + \left[\eta(s) - \eta(s_o)\right]^2} ds = F(s_o) + \pi k(s_o).$$

This is an integral equation for the determination of $k(s)$. Divide through by π and put

$$(52) \quad \widetilde{K}(s_o, s) = \frac{1}{\pi} \frac{\left[\eta(s) - \eta(s_o)\right]\xi'(s_o) - \left[\xi(s) - \xi(s_o)\right]\eta'(s_o)}{\left[\xi(s) - \xi(s_o)\right]^2 + \left[\eta(s) - \eta(s_o)\right]^2}$$

and

$$f(s_o) = -\frac{F(s_o)}{\pi}.$$

Then (51) becomes

$$(53) \qquad k(s_o) = f(s_o) + \int_0^l \widetilde{K}(s_o, s)k(s)ds.$$

Introduce a parameter λ and write

$$(54) \qquad k(s_o) = f(s_o) + \lambda \int_0^l \widetilde{K}(s_o, s)k(s)ds,$$

which becomes identical with (53) for $\lambda = +1$.

From (51) we see that $k(0) = k(l)$, since, from our hypothesis, $F(0) = F(l)$ and $\widetilde{K}(0, s) = \widetilde{K}(l, s)$.

Second Method.—From Theorem XXII we have

$$(55)\qquad \begin{aligned} \frac{\partial \mathrm{v}_i}{\partial n_i} &= -\pi k(s_o) + \int_0^l k(s)\frac{\cos(r_o n_i)}{r_o}ds \\ \frac{\partial \mathrm{v}_e}{\partial n_i} &= +\pi k(s_o) + \int_0^l k(s)\frac{\cos(r_o n_i)}{r_o}ds. \end{aligned}$$

We now seek, as a solution of Neumann's problem, a function u which is harmonic everywhere except on C, and which on C satisfies the condition

$$(56)\qquad h\frac{\partial u_i}{\partial n_i} + \frac{\partial u_e}{\partial n_i} = hF(s_o) + G(s_o),$$

where h is an arbitrary parameter. For $h = \infty$, we have the so-called interior case, while for $h = 0$ we have the exterior case. This condition is more general than (50) and for $h = \infty$ reduces to (50).

The function $\mathrm{v}(x, y)$ is harmonic everywhere except on C. This function will then be a solution of Neumann's problem provided $k(s)$ can be so determined that $u = \mathrm{v}(x, y)$ satisfies (56). Substitute now from (55) in (56) for

$$\frac{\partial \mathrm{v}_i}{\partial n_i} = \frac{\partial u_i}{\partial n_i} \text{ and } \frac{\partial \mathrm{v}_e}{\partial n_i} = \frac{\partial u_e}{\partial n_i}.$$

We find

$$(57)\qquad k(s_o) = f(s_o) + \lambda\int_0^l k(s)\frac{\cos(r_o n_i)}{\pi r_o}ds,$$

where $f(s_o) = \dfrac{G(s_o) + hF(s_o)}{\pi(1 - h)}$, $\lambda = \dfrac{h + 1}{h - 1}$.

This is an integral equation of the second kind with a parameter λ. The kernel is $\dfrac{\cos(r_o n_i)}{\pi r_o} \equiv \widetilde{K}(s_o, s)$. For $h = 0$, we have the exterior problem, but for $h = 0$ we have $\lambda = -1$. For $h = \infty$, we have the interior problem, but for $h = \infty$ we have $\lambda = +1$ and (57) reduces to (53).

52. Solution of the Integral Equation.—To the equation (54) the Fredholm theory which we have developed can be applied. We look, then, for a solution of (54) when $\lambda = +1$. Now $\lambda = +1$ is a characteristic constant of $\widetilde{K}(s_o, s)$, for $\widetilde{K}(s_o, s)$ can be obtained from the kernel $K(s_o, s)$ for Dirichlet's problem by interchanging s and s_o. Thus $\widetilde{K}$ and K are adjoint kernels. Then from Theorem IX, Chap. III, $\widetilde{K}(s_o, s)$ and $K(s_o, s)$ have the same characteristic constants with the same indices. But $\lambda = +1$ has been shown by Theorem XIX to be a characteristic constant of index 1 for $K(s_o, s)$. Therefore $\lambda = +1$ is a characteristic constant of index 1 for $\widetilde{K}(s_o, s)$. According to Fredholm's third fundamental theorem, (54) has, in general, no solution for $\lambda = +1$. But if $f(s_o)$ satisfies certain $q = 1$ conditions, then there will be a solution. This condition is

$$\int_0^l f(s)\bar{\varphi}(s)ds = 0, \tag{58}$$

where $\bar{\varphi}(s_o)$ is a solution of the associated homogeneous integral equation

$$\bar{\varphi}(s_o) = \int_0^l K(s_o, s)\bar{\varphi}(s)ds, \tag{59}$$

the kernel of which is the adjoint of the kernel of the integral equation (54) under consideration. But $\lambda = +1$ is a characteristic constant of $K(s_o, s)$ of index 1 and $\bar{\varphi}(s) \equiv 1$ is a solution of (59), as shown in §47. Therefore, condition (58) becomes

$$\int_0^l f(s)ds = 0. \tag{60}$$

Unless this condition is satisfied, there is no solution. If this condition is satisfied, then there are ∞^1 solutions. By referring to the definition of $f(s)$, we see that (60) gives the condition

$$\int_0^l F(s)ds = 0.$$

Thus we see that $v(x, y)$ is a solution of Neumann's problem. Furthermore, by referring to Theorems XX, XXI, and XXII, we see that $v(x, y)$ satisfies the conditions (A) of §38. Let $\bar{v}(x, y)$ be any other solution of Neumann's problem which satisfies the conditions (A). Form the difference

$$\varpi(x, y) = \bar{v}(x, y) - v(x, y).$$

Then it follows that

1) $\Delta\varpi = 0$ on I'.

2) $\dfrac{\partial \varpi_i}{\partial n_i} = 0$ along C.

From Corollary II to Theorem XI it follows that

$$\varpi = C \text{ (a constant) on } I'$$

Therefore,

$$\bar{v} = v + c.$$

Hence, there are ∞^1 solutions of Neumann's problem which differ only by an additive constant. Furthermore, these are the only solutions of the problem which satisfy the conditions (A). We have now the following theorem:

Theorem XXIII.—*Given a closed curve C satisfying the conditions* 1) . . . 4) *of* §36

C: $$x = \xi(s), \quad y = \eta(s), \quad 0 \leqq s \leqq l$$
$$\xi(0) = \xi(l), \eta(0) = \eta(l),$$

and the function $F(s)$ continuous on C, $F(0) = F(l)$, and

(61) $$\int_0^l F(s)ds = 0,$$

then Neumann's problem has ∞^1 solutions

$$\bar{u} = u + c$$

where

$$u = \int_0^l k(s) \log \frac{1}{r}\, ds$$

and $k(s)$ *is a solution of*

$$k(s_o) = f(s_o) + \lambda \int_0^l \widetilde{K}(s_o, s) k(s) ds$$

for $\lambda = 1$, *where* $\widetilde{K}(s_o, s)$ *is defined by* (52). *If the condition* (61) *is not satisfied, the problem has no solution.*

CHAPTER V

HILBERT-SCHMIDT THEORY OF INTEGRAL EQUATIONS WITH SYMMETRIC KERNELS

Solution Expressed in Terms of a Set of Fundamental Functions

I. Existence of at Least One Characteristic Constant

53. Introductory Remarks.—The Fredholm theory of a linear integral equation with a parameter λ:

$$(1) \qquad u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$$

has been developed under the assumptions

$A)$ $K(x, t)$ real.
$B)$ $K(x, t)$ continuous.
$C)$ $K(x, t) \not\equiv 0$ on $R : a \leq x \leq b, a \leq t \leq b$.

For the Hilbert-Schmidt theory a fourth assumption is made that the kernel is symmetric.

$$(2) \quad D) \ K(x, t) = K(t, x).$$

It is clear that the results of the Fredholm theory still hold. But, besides, a number of new results will follow from the additional condition D). These results were first obtained by Hilbert.

Fredholm obtained the solution of (1) by considering it as the limit of a set of linear equations. He did not carry through the limiting processes, but guessed at the solution and then showed independently of the linear equations that the guess was correct. Hilbert started from the finite system of linear equations, of which equations (1) may be considered as a limit, and actually carried through the limiting process in detail. E. Schmidt, in 1906, obtained Hilbert's results directly without using the limiting process. Hilbert and Schmidt do not make use of Fredholm's results. We shall, however, avail ourselves of these results whenever they lead to simplifications of the proofs.

The first fundamental theorem of Hilbert's theory is the following.

Theorem I. *Every symmetric kernel has at least one characteristic constant (real or imaginary).*

Our proof of this theorem involves several lemmas which we will give first.

We give an example due to Kowalewski, which illustrates that a kernel which is not symmetric does not necessarily have a characteristic constant. The kernel

$$K(x, t) = \sin (\pi x) \cos (\pi t), \quad [01]$$

has no characteristic constants, if

$$(3) \qquad u(x) = \lambda \int_0^1 \sin (\pi x) \cos (\pi t) u(t) dt$$

has no solution other than $u \equiv 0$.

Equation (3) may be written

$$u(x) = \lambda \sin \pi x \int_0^1 \cos (\pi t) u(t) dt \equiv c \sin \pi x,$$

where c is a constant yet to be determined. Substitute this expression for $u(x)$ in (3). We obtain

$$c \sin \pi x = c\lambda \sin \pi x \int_0^1 \cos (\pi t) \sin (\pi t) dt.$$

But
$$\int_0^1 \cos (\pi t) \sin (\pi t)\, dt = 0.$$

Therefore $\qquad c \sin \pi x \equiv 0,$

whence $\qquad c = 0,$

and $\qquad u \equiv 0.$

Thus this particular kernel has no characteristic constants. This shows that Theorem I indeed states a property peculiar to *symmetric* kernels.

54. Power Series for $\frac{D'(\lambda)}{D(\lambda)}$.—The following proof of the theorem is due to Kneser. It is based upon a lemma concerning the expansion of $\frac{D'(\lambda)}{D(\lambda)}$, which holds under the assumptions *A*), *B*), *C*).

From Fredholm's first fundamental theorem, if $D(\lambda) \neq 0$, then (1) has one and only one solution given by

$$u(x) = f(x) + \int_a^b \frac{D(x, t; \lambda)}{D(\lambda)} f(t)dt. \tag{4}$$

Consider the expansion of $\frac{D(x, t; \lambda)}{D(\lambda)}$ as a power series in λ. Now $D(0) = 1 \neq 0$ and, therefore, there exists a ρ such that if $|\lambda| < \rho$, then

$$D(\lambda) \neq 0, \quad |\lambda| < \rho. \tag{5}$$

Hence, since $D(\lambda)$ is permanently convergent, it can be expanded[1] into a power series convergent for $|\lambda| < \rho$. Then

$$\frac{1}{D(\lambda)} = d_0 + d_1\lambda + d_2\lambda^2 + \cdots, \quad |\lambda| < \rho. \tag{6}$$

Now, $D(x, t; \lambda)$ is a permanently converging power series in λ. Hence, also, the product $D(x, t; \lambda)\cdot\frac{1}{D(\lambda)}$ can be expanded into a power series in λ:

$$\frac{D(x, t; \lambda)}{D(\lambda)} = \sum_{n=1}^{\infty} g_n(x, t)\lambda^n \tag{7}$$

[1] HARKNESS and MORLEY, "Theory of Functions," §83.

and the series will be uniformly convergent as to x and t in R for $|\lambda| < \rho$. The right-hand member of (7) has no term free from λ for

$$D(x, t; \lambda) = \lambda K(x, t) + \ . \ . \ .$$

Substitute from (7) in (4) and we obtain

$$(8) \qquad u(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b g_n(x, t) f(t) dt, \ |\lambda| < \rho.$$

We have been able to interchange the order of the integration and summation in (8) on account of the uniform convergence of $\sum_{n=1}^{\infty} g_n(x, t)\lambda^n$.

The coefficients $g_n(x, t)$ can be determined by comparing (8) with the expression for the solution obtained by successive approximations (see §8):

$$(9) \qquad u(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x, t) f(t) dt$$

where $K_n(x, t)$ are the iterated kernels given by

$$(10) \qquad K_n(x, t) = \int_a^b K(x, s) K_{n-1}(s, t) ds.$$

This solution by successive approximations is valid if $|\lambda| < \frac{1}{M(b - a)}$, where M is the maximum of $|K(x, t)|$ on R. Denote by r the smaller of the two quantities ρ, $\frac{1}{M(b - a)}$. Then (8) and (9) hold simultaneously for $|\lambda| < r$. But (1) has one and only one solution; therefore (8) and (9) represent the same function, and coefficients of corresponding powers of λ must be equal. Therefore,

$$\int_a^b g_n(x, t) f(t) dt = \int_a^b K_n(x, t) f(t) dt, \ (x)$$

where the notation (x) is used to mean uniformly as to x. Hence

$$(11) \qquad \int_a^b \left\{ K_n(x, t) - g_n(x, t) \right\} f(t)dt = 0, \ (x).$$

But $K_n(x, t)$ and $g_n(x, t)$ are independent of f and, for a given value of x,

$$K_n(x, t) - g_n(x, t) = M(t)$$

is a real and continuous function of t. Now (11) holds for any choice of f which is continuous. Choose then $f(t) = M(t)$. Then (11) becomes

$$\int_a^b \left[M(t) \right]^2 dt = 0.$$

Hence

$$M(t) \equiv 0 \text{ on } \left[ab\right].$$

Therefore

$$g_n(x, t) \equiv K_n(x, t), \quad (x, t)$$

and (7) becomes

$$(12) \qquad \frac{D(x, t; \lambda)}{D(\lambda)} = \sum_{n=1}^{\infty} K_n(x, t)\lambda^n, \ |\lambda| < r.$$

Thus, under the assumptions A), B), C), we have proved the following theorem:

Theorem II.—*For all sufficiently small values of* λ

$$\frac{D(x, t; \lambda)}{D(\lambda)} = \sum_{n=1}^{\infty} K_n(x, t)\lambda^n,$$

the series converging uniformly as to x and t in R.

Theorem II holds when $t = x$. Then

$$\frac{D(x, x; \lambda)}{D(\lambda)} = \sum_{n=1} K_n(x, x)\lambda^n, \ |\lambda| < r$$

and $\sum K_n(x, x)\lambda^n$ is uniformly convergent on $\left[ab\right]$

Integrate this equality with respect to x from a to b. We obtain

$$\frac{1}{D(\lambda)}\int_a^b D(x, x; \lambda)dx = \sum_{n=1}^{\infty}\lambda^n \int_a^b K_n(x, x)dx.$$

Let us introduce the permanent notation

(13) $$\int_a^b K_n(x, x)dx = U_n, \text{ a constant,}$$

and recall from Chap. III, §20, equation (27) that

$$\int_a^b D(x, x; \lambda) = -\lambda D'(\lambda).$$

Then our equality above becomes

$$-\frac{\lambda D'(\lambda)}{D(\lambda)} = \sum_{n=1}^{\infty} U_n\lambda^n,$$

which may be written

$$\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=0}^{\infty} U_{n+1}\lambda^n, \ |\lambda|<r.$$

We have thus the following corollary to Theorem II:

Corollary.—

(14) $$\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=0}^{\infty} U_{n+1}\lambda^n, \ |\lambda|<r.$$

55. Plan of Kneser's Proof.—Suppose that $K(x, t)$ has no characteristic constant; that is, that $D(\lambda) = 0$ has no roots, real or imaginary. Then the quotient $\frac{D'(\lambda)}{D(\lambda)}$ can be directly expanded into a power series

$$\frac{D'(\lambda)}{D(\lambda)} = \sum_{n=0}^{\infty} C_n\lambda^n,$$

and this series will be permanently converging.[1] But (14) holds for $|\lambda| < r$. Then

$$\sum_{n=0}^{\infty} C_n \lambda^n = \sum_{n=0}^{\infty} U_{n+1}\lambda^n, \ |\lambda| < r,$$

hence

$$C_n = U_{n+1}$$

and, therefore,

$$\sum_{n=0}^{\infty} U_{n+1}\lambda^n \text{ is permanently convergent.}$$

Then also[2] is

$$\sum_{n=0}^{\infty} |U_{n+1}||\lambda|^n \ \text{ permanently convergent,}$$

and also

(15) $$\sum_{n=0}^{\infty} |U_{n+1}||\lambda|^{n+1} \text{ is permanently convergent.}$$

Then the series formed by omitting any number of terms from (15) is also permanently convergent. Hence

(16) $$\sum_{n=0}^{\infty} |U_{2n}||\lambda|^{2n} \text{ is permanently convergent.}$$

We have proved that (16) was permanently convergent by assuming that $K(x, t)$ had no characteristic constant. If, then, for a given kernel $K(x, t)$ we can show that the corresponding series (16) is *not* permanently convergent, we will have shown that $K(x, t)$ has at least one characteristic constant. We are then going to prove that if, in addition to the properties $A)$ $B)$ $C)$, the kernel $K(x, t)$ is symmetric, then the corresponding series (16) is not permanently convergent. Theorem I will then be proved.

[1] Harkness and Morley, *Loc. cit.*, §83.
[2] Harkness and Morley, *Loc. cit.*, §76.

For this purpose we need some auxiliary lemmas on iterated kernels.

56. Lemmas on Iterations of a Symmetric Kernel.—The iterated kernel $K_n(x, t)$ was defined by

$$(17)\qquad K_1(x, t) = K(x, t)$$
$$K_n(x, t) = \int_a^b K(x, s)K_{n-1}(s, t)ds.$$

By successive applications of this recursion formula we find

$$(18)\quad K_n(x, t) = \int_a^b \cdots \int_a^b K(x, s_1)K(s_1, s_2) \cdots K(s_{n-1}, t)ds_1 \cdots ds_{n-1}.$$

In Chap. II, §11, we showed that

$$(19)\qquad K_{n+p}(x, t) = \int_a^b K_n(x, s)K_p(s, t)ds.$$

We now establish the following lemmas:

Lemma 1.—If $K(x, t)$ is symmetric, then $K_n(x, t)$ is symmetric.

Proof.—By (18) we have

$$K_n(t, x) = \int_a^b \cdots \int_a^b K(t, s_1)K(s_1, s_2) \cdots K(s_{n-1}, x)\, ds_1 \cdots ds_{n-1}$$
$$= \int_a^b \cdots \int_a^b K(s_1, t)K(s_2, s_1) \cdots K(x, s_{n-1})\, ds_1 \cdots ds_{n-1}$$

on account of the symmetry of $K(x, t)$

$$= \int_a^b \cdots \int_a^b K(x, s_{n-1}) \cdots K(s_2, s_1)K(s_1, t)\, ds_1 \cdots ds_{n-1}.$$

Now make a change of notation.

For $\quad s_1 \quad s_2 \quad \cdots \quad s_{n-2} \quad s_{n-1}$

put $\quad s_{n-1} \quad s_{n-2} \quad \cdots \quad s_2 \quad s_1.$

Then

$$K_n(t, x) = \int_a^b \cdots \int_a^b K(x, s_1)K(s_1, s_2) \cdots K(s_{n-1}, t)\, ds_1 \cdots ds_{n-1}$$

$$= K_n(x, t) \text{ by (18).}$$

Lemma II.—If $K(x, t)$ satisfies the conditions A) B) C) D), then $K_n(x, t) \not\equiv 0$ on R.

Proof.—Suppose $K_n(x, t) \equiv 0$ on R, and suppose that n is the lowest index for which an iterated kernel vanishes identically:

$$K(x, t) \not\equiv 0, K_2(x, t) \not\equiv 0, \ldots, K_{n-1}(x,t) \not\equiv 0,$$
$$K_n(x, t) \equiv 0, R.$$

Then certainly, by C), we have $n > 1$. Then from (19) it follows that all following iterated kernels are identically zero. Then

$$K_{n+1}(x, t) \equiv 0 \text{ on } R.$$

Now either n or $n + 1$ must be an even number $2m$:

$$2m = n, \text{ or } n + 1.$$

Then

$$K_{2m}(x, t) \equiv 0 \text{ on } R.$$

But by (19)

$$K_{2m}(x, t) = \int_a^b K_m(x, s)K_m(s, t)ds.$$

This equality holds when $t = x$. Then

$$K_{2m}(x, x) = \int_a^b K_m(x, s)K_m(s, x)ds$$

(20) $$= \int_a^b \left[K_m(x, s) \right]^2 ds$$

on account of the symmetry of $K(x, t)$.
But $K_{2m}(x, t) \equiv 0$ on R. Therefore, since $K_m(x, t)$ is a real function of x and t, we have

$$K_m(x, t) \equiv 0 \text{ on } R.$$

But $$m = \frac{n}{2}, \text{ or } \frac{n+1}{2}.$$

Hence $$n - m = \frac{n}{2}, \text{ or } \frac{n-1}{2} > 0 \text{ for } n > 1.$$

Therefore $$m > n. \text{ ?}$$

Thus we have contradicted the assumption that $K_n(x, t)$ was the first iterated function to vanish and the lemma is proved.

It is now clear that if $K(x, t)$ satisfies the conditions A) B) C) D), then $K_n(x, t)$ satisfies the same conditions.

Corollary.— $$U_{2m} > 0.$$

Proof.— $$U_{2m} = \int_a^b K_{2m}(x, x)dx.$$

Now $K_{2m}(x, x) > 0$ by (20), since $K_m(x, t) \not\equiv 0$.

57. Schwarz's Inequality.—Let $\varphi(x)$ and $\psi(x)$ be real and continuous on the interval $[ab]$, and u and v any two reals constant with respect to x. Then

$$\int_a^b \left[u\varphi(x) + \text{v}\psi(x) \right]^2 dx \geqq 0,$$

since we have the square of a real function.

Expand and we obtain

$$u^2 \int_a^b \varphi^2(x)dx + 2u\text{v} \int_a^b \varphi(x)\psi(x)dx + \text{v}^2 \int_a^b \psi^2(x)dx \geqq 0.$$

This is a definite quadratic form and hence

$$\text{(21)} \quad \left[\int_a^b \varphi(x)\psi(x)dx \right]^2 \leqq \int_a^b \varphi^2(x)dx \times \int_a^b \psi^2(x)dx.$$

This is called the inequality of Schwarz. In like manner, we can prove that if $\varphi(x, y)$ and $\psi(x, y)$ are real and continuous on R in x and y, then

$$\left[\int_a^b \int_a^b \varphi(x, y)\psi(x, y)dxdy \right]^2 \leqq \left[\int_a^b \int_a^b \varphi^2(x, y)dxdy \right] \left[\int_a^b \int_a^b \psi^2(x, y)dxdy \right].$$

58. Application of Schwarz's Inequality.—We have obtained one expression for U_{2n}:

(13) $$U_{2n} = \int_a^b K_{2n}(x, x)dx$$

(22) $$= \int_a^b \int_a^b \Big[K_n(x, t)\Big]^2 dtdx > 0 \text{ by Lemma II.}$$

We now proceed to determine a second expression for U_{2n}. Now by (19) we have

$$K_{2n}(x, t) = \int_a^b K_{n-1}(x, s)K_{n+1}(s, t)dt.$$

If we put $t = x$, this equality becomes

$$K_{2n}(x, x) = \int_a^b K_{n-1}(x, t)K_{n+1}(t, x)dt$$

$$= \int_a^b K_{n-1}(x, t)K_{n+1}(x, t)dt,$$

on account of the symmetry of $K_{n+1}(x, t)$. Then

(23) $$U_{2n} = \int_a^b \int_a^b K_{n-1}(x, t)K_{n+1}(x, t)dtdx.$$

Apply Schwarz's inequality to the right member of (23). We obtain

$$\left[\int_a^b \int_a^b K_{n-1}(x, t)K_{n+1}(x, t)dtdx\right]^2$$
$$\leqq \left[\int_a^b \int_a^b K^2{}_{n-1}(x, t)dtdx\right]\left[\int_a^b \int_a^b K^2{}_{n+1}(x, t)dtdx\right],$$

which, if we make use of (22) and (23), may be written

$$U^2{}_{2n} \leqq U_{2n-2} \cdot U_{2n+2}.$$

Divide both members of this inequality by $U_{2n-2} \cdot U_{2n}$. This is possible, since $U_{2n-2} \neq 0$, $U_{2n} \neq 0$. We obtain

(24) $$\frac{U_{2n+2}}{U_{2n}} \geqq \frac{U_{2n}}{U_{2n-2}}.$$

Putting successively $n = 2, 3, \ldots, n$, we obtain in this way the sequence of inequalities

$$\frac{U_{2n+2}}{U_{2n}} \geqq \frac{U_{2n}}{U_{2n-2}} \geqq \cdots \geqq \frac{U_6}{U_4} \geqq \frac{U_4}{U_2}.$$

Therefore

(25) $$\frac{U_{2n+2}}{U_{2n}} \geqq \frac{U_4}{U_2}, \ (n).$$

Apply now the ratio test to the series

(16) $$\sum_{n=1}^{\infty} U_{2n}|\lambda|^{2n} \equiv \sum_{n=1}^{\infty} V_n$$

to find the radius of convergence in λ. We find

$$\frac{V_{n+1}}{V_n} = \frac{U_{2n+2}}{U_{2n}}|\lambda|^2 \geqq \frac{U_4}{U_2}|\lambda|^2, \ (n).$$

Therefore the series diverges if

$$\frac{U_4}{U_2}|\lambda|^2 > 1,$$

that is, if

$$|\lambda| > \sqrt{\frac{U_2}{U_4}}.$$

Therefore, the series is not a permanently convergent power series in λ. This completes the proof of Theorem I.

Theorem I.—*For a real, symmetric, continuous, non-identically vanishing kernel $K(x, t)$ there exists at least one characteristic constant λ_o.*

We remark that for every value of λ in the λ-plane, without a circle C of radius $\sqrt{\frac{U_2}{U_4}}$, $\sum_{n=1}^{\infty} U_{2n}|\lambda|^{2n}$ diverges. Hence, from (14) and (16), $\frac{D'(\lambda)}{D(\lambda)}$, when expressed as a series in λ, diverges for all values of λ without the circle C. Then, certainly, $D(\lambda) = 0$ has at least one root within the interior or on the boundary of C

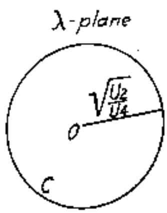


FIG. 18.

II. Orthogonality

59. Orthogonality Theorem.—From the Fredholm theory

1) Two associated kernels $\widetilde{K}(x, t)$, $K(x, t)$:

$$\widetilde{K}(x, t) = K(t, x)$$

have the same characteristic constants with the same indices; if, then

2) λ_o and λ_1 $(\lambda_o \neq \lambda_1)$ are two characteristic constants of $K(x, t)$, and hence of $K(t, x)$, and $\varphi_o(x)$ is a fundamental function of $K(x, t)$ for λ_o, and $\tilde{\varphi}_1(x)$ is a fundamental function of $\widetilde{K}(x, t)$ for λ_1, it follows that $\int_a^b \varphi_o(x)\tilde{\varphi}_1(x)dx = 0$.

But if the kernel $K(x, t)$ is symmetric, then $\tilde{\varphi}_1(x)$ is also a fundamental function of $K(x, t)$ for λ_1. If, then, we write $\tilde{\varphi}_1 \equiv \varphi_1(x)$, we have

$$\int_a^b \varphi_o(x)\varphi_1(x)dx = 0.$$

Definition.—Two functions $\varphi(x)$, $\psi(x)$, continuous on the interval $[ab]$ are said to be orthogonal on $[ab]$ if

$$(\varphi\psi) = \int_a^b \varphi(x)\psi(x)dx = 0.$$

We have thus proved the following theorem which is called the *orthogonality theorem:*

Theorem III.—*If $K(x, t)$ is symmetric and $\varphi_o(x)$, $\varphi_1(x)$ are fundamental functions of $K(x, t)$ for λ_o and λ_1 respectively $(\lambda_o \neq \lambda_1)$, then $\varphi_o(x)$ and $\varphi_1(x)$ are orthogonal on the interval $[ab]$.*

$$\int_a^b \varphi_o(x)\varphi_1(x)dx = 0. \tag{26}$$

As an illustration of Theorem III, take the symmetric kernel $K(x, t)$:

$$K(x, t) = \begin{cases} (1 - t)x, & [0t] \\ (1 - x)t, & [t1] \end{cases},$$

which has the characteristic constants

$$\lambda_n = n^2\pi^2, \quad n = 1, 2, \ldots$$

and the corresponding fundamental functions

$$\varphi_n(x) = \sin n\pi x.$$

It is then well known from the theory of definite integrals that

$$\int_0^1 \sin n\pi x \sin p\pi x dx = 0, \quad (n \neq p). \tag{27}$$

60. Reality of the Characteristic Constants.—By means of the preceding theorem we can now prove the following theorem due to Hilbert:

Theorem IV.—*If $K(x, t)$ is real and symmetric, continuous, and $\not\equiv 0$, then all of the characteristic constants are real.*

Proof.—Suppose that there is a characteristic constant λ_o not real:

$$\lambda_o = \mu_o + i\nu_o, \quad \nu_o \neq 0.$$

Then the homogeneous equation

$$u(x) = \lambda_o \int_a^b K(x, t)u(t)dt$$

has at least one continuous solution $\varphi(x) \equiv 0$:

$$\varphi(x) = (\mu_o + i\nu_o) \int_a^b K(x, t)\varphi(t)dt. \tag{28}$$

1) Suppose $\varphi(x)$ real, then separating the real and imaginary parts of (28), we obtain

$$\varphi(x) = \mu_o \int_a^b K(x, t)\varphi(t)dt \tag{29}$$

$$0 = \nu_o \int_a^b K(x, t)\varphi(t)dt. \tag{30}$$

From (30)

$$\int_a^b K(x, t)\varphi(t)dt = 0,$$

since $\nu_o \neq 0$. Then from (29), $\varphi(x) \equiv 0$, which contradicts the assumption that $\varphi(x) \not\equiv 0$. Therefore, $\varphi(x)$ cannot be real.

2) Suppose $\varphi(x) = \text{v}(x) + iw(x)$, where $\text{v}(x)$ and $w(x)$ are real. Then (28) becomes

$$(31)\quad \text{v}(x) + iw(x) = (\mu_o + i\nu_o)\int_a^b K(x,t)\Big[\text{v}(t) + iw(t)\Big]dt.$$

Separate (31) into its real and imaginary parts. We obtain

$$(32)\qquad \text{v}(x) = \mu_o\int_a^b K(x,t)\text{v}(t)dt - \nu_o\int_a^b K(x,t)w(t)dt$$

$$(33)\quad w(x) = \mu_o\int_a^b K(x,t)w(t)dt + \nu_o\int_a^b K(x,t)\text{v}(t)dt.$$

Multiply both sides of (33) by $-i$ and add to (32). We obtain

$$\text{v}(x) - iw(x) = (\mu_o - i\nu_o)\int_a^b K(x,t)\Big[\text{v}(t) - iw(t)\Big]dt.$$

Therefore $\text{v}(x) - iw(x) \equiv \bar{\varphi}(x)$ is also a fundamental function of $K(x, t)$, belonging to $\bar{\lambda}_0 \equiv \mu_o - i\nu_o$. If we now apply the orthogonality Theorem III, we obtain, since $\bar{\lambda}_0 \neq \lambda_0$,

$$\int_a^b \varphi(x)\bar{\varphi}(x)dx = 0,$$

which may be written

$$\int_a^b \Big[v^2(x) + w^2(x)\Big]dx = 0.$$

But $v(x)$ and $w(x)$ are real functions. Therefore

$$v^2(x) + w^2(x) = 0,\ (x)$$

and hence $$v(x) \equiv 0,\ w(x) \equiv 0$$

and therefore $$\varphi(x) \equiv 0,$$

which constitutes a contradiction. Therefore λ_o cannot be of the form $\lambda_o = \mu_o + i\nu_o(\nu_o \neq 0)$ and hence λ_o *must be real.*

This result might have been foreseen from the analogy with the finite system of linear equations:

$$u(t_i) - \lambda h \sum_{j=1}^{n} K(t_i, t_j)u(t_j) = f(t_i) \quad (i = 1, \ . \ . \ . \ , n)$$

with determinant

$$(34) \quad \Delta = \begin{vmatrix} 1 - \lambda h K_{11} & - \lambda h K_{12} & \cdots & - \lambda h K_{1n} \\ - \lambda h K_{21} & 1 - \lambda h K_{22} & \cdots & - \lambda h K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ - \lambda h K_{n1} & - \lambda h K_{n2} & \cdots & 1 - \lambda h K_{nn} \end{vmatrix}.$$

But it is well known[1] that, if K_{ij} is real and $K_{ij} = K_{ji}$, then all of the roots of (34) are real.

In the Fredholm theory we proved that $q \leqq r$ where q is the index of λ_o and r is the multiplicity of the root λ_o of $D(\lambda) = 0$. For the case of a symmetric kernel, Hilbert has proved[2] the following more definite theorem which we state here without proof:

Theorem V.—*For a real symmetric kernel the index q of a characteristic constant λ_o is always equal to its multiplicity r: $q = r$.*

61. Complete Normalized Orthogonal System of Fundamental Functions.—Let us suppose that λ_o is a characteristic constant of the symmetric kernel $K(x, t)$ of index q. Then the equation

$$u(x) = \lambda_o \int_a^b K(x, t)u(t)dt$$

has been shown to have q linearly independent solutions:

$$\varphi_\alpha(x) = \frac{D\begin{pmatrix} x_1', & \cdots, & x'_{\alpha-1}, & x\ , & x'_{\alpha+1}, & \cdots, & x_q' \\ y_1', & \cdots, & y'_{\alpha-1}, & y_\alpha', & y'_{\alpha+1}, & \cdots, & y_q' \end{pmatrix}\lambda_o}{D\begin{pmatrix} x_1', & \cdots, & x'_{\alpha-1}, & x_\alpha', & x'_{\alpha+1}, & \cdots, & x_q' \\ y_1', & \cdots, & y'_{\alpha-1}, & y_\alpha', & y'_{\alpha+1}, & \cdots, & y_q' \end{pmatrix}\lambda_o}$$

$$\alpha = 1, \ . \ . \ . \ , q.$$

[1] See Bôcher, "Introduction to Higher Algebra," §59.

[2] For proof, see Horn, §39.

For a real symmetric kernel, λ_o and, therefore, also $\varphi_\alpha(x)$ are real, since the arguments in the numerator and denominator of the expression for $\varphi_\alpha(x)$ are all real. Any other solution $\psi_\alpha(x)$ is expressible linearly in terms of these q solutions, or in terms of q linearly independent linear combinations of the $\varphi_\alpha(x)$'s:

(35) $$\psi_\alpha(x) = c_{\alpha 1}\varphi_1(x) + \ . \ . \ . \ + c_{\alpha q}\varphi_q(x),$$

where the determinant of the coefficients $|c_{\alpha\beta}| \neq 0$, $\left[\alpha, \beta = 1, \ . \ . \ . \ , q\right]$ does not vanish.

We are going to choose such a system of fundamental functions in such a way that they form what is called a *normalized orthogonal system.*

a) Normalized Fundamental Functions.—A function ψ is said to be *normalized* if

$$(\psi\psi) \equiv \int_a^b \psi^2(x)dx = 1.$$

If $\varphi(x)$ is a fundamental function belonging to λ_o:

$$\varphi(x) = \lambda_o \int_a^b K(x, t)\varphi(t)dt, \quad \left[\varphi(x) \neq 0\right]$$

then $\psi(x) = c\varphi(x)$ $(c \neq 0)$ is clearly again a fundamental function belonging to λ_o, and we can choose c so that

(36) $$(\psi\psi) = 1,$$

that is

$$c^2 \int_a^b \varphi^2(x)dx = 1,$$

whence

(37) $$c = \pm \frac{1}{\sqrt{\int_a^b \varphi^2(x)dx}}.$$

This value for c is finite, real, and $\neq 0$. Hence we obtain

$$\psi(x) = \frac{\varphi(x)}{\sqrt{\int_a^b \varphi^2(x)dx}}. \tag{38}$$

b) Normalized Orthogonal Systems.—We now propose to determine the constants $c_{\alpha\beta}$ in (35) in such a way that the ψ_α's satisfy the following two conditions:

1) All ψ_α's are normalized:

$$\int_a^b \psi_\alpha^2(x)dx = 1, \alpha = 1, \ . \ . \ . \ , q \tag{39}$$

2) Two ψ_α's with different subscripts are orthogonal:

$$\int_a^b \psi_\alpha(x)\psi_\beta(x)dx = 0,\ \alpha \neq \beta,\ (\alpha, \beta = 1, \ . \ . \ . \ , q). \tag{40}$$

For this purpose we put $\psi_1(x) = c\varphi_1(x)$, where

$$c = \frac{1}{\sqrt{\int_a^b \varphi_1^2(x)dx}}.$$

Then the condition $\int_a^b \psi_1^2(x)dx = 1$ is satisfied.

Next choose $\psi_2(x)$ as a linear function of φ_1 and φ_2. On account of (38) it can then be expressed linearly in terms of ψ_1 and φ_2:

$$\psi_2(x) = \alpha_1\psi_1(x) + \alpha_2\varphi_2(x).$$

We now determine α_1 and α_2 so that the conditions 1) and 2) are satisfied for ψ_1 and ψ_2.

From (40)

$$(\psi_1\, \psi_2) = \alpha_1\int_a^b \psi_1^2 dx + \alpha_2\int_a^b \psi_1(x)\varphi_2(x)dx = 0. \tag{42}$$

Therefore, on account of (39),

$$\alpha_1 = -\alpha_2(\psi_1\, \varphi_2)$$

and (41) becomes

$$\psi_2(x) = \alpha_2\left[\varphi_2(x) - (\psi_1\varphi_2)\psi_1(x)\right].$$

We can now determine α_2 in such a way that $\psi_2(x)$ is normalized provided

$$\varphi_2(x) - (\psi_1\varphi_2)\psi_1(x) \not\equiv 0. \tag{43}$$

But $(\psi_1\varphi_2)$ is a constant and if (43) is not true, then

$$\varphi_2(x) + c_1\varphi_1(x) \equiv 0$$

which is a linear relation between two fundamental functions φ_1 and φ_2 and, therefore, φ_1 and φ_2 would be linearly dependent, which is contrary to hypothesis. Therefore, $\psi_2(x)$ is completely determined as a linear function of φ_1 and φ_2 or of ψ_1 and φ_2. Now choose $\psi_3(x)$ as a linear function of $\varphi_1, \varphi_2, \varphi_3$, or, what amounts to the same thing, as a linear function of $\psi_1, \psi_2, \varphi_3$:

$$\psi_3(x) = \beta_1\psi_1 + \beta_2\psi_2 + \beta_3\varphi_3.$$

Apply condition (2). We obtain

$$(\psi_1\psi_3) = \beta_1(\psi_1\psi_1) + \beta_2(\psi_1\psi_2) + \beta_3(\psi_1\varphi_3) = 0.$$

But $(\psi_1\psi_1) = 1$ and $(\psi_1\psi_2) = 0$, therefore

$$(\psi_1\psi_3) = \beta_1 + \beta_2(\psi_1\varphi_3) = 0.$$

Hence

$$\beta_1 = -\beta_3(\psi_1\varphi_3).$$

Also by condition (2) we have

$$(\psi_2\psi_3) = \beta_1(\psi_2\psi_1) + \beta_2(\psi_2\psi_2) + \beta_3(\psi_2\varphi_3) = 0,$$

from which we derive

$$\beta_2 = -\beta_3(\psi_2\varphi_3).$$

The expression for $\psi_3(x)$ now becomes

$$\psi_3(x) = \beta_3\left[\varphi_3 - (\psi_1\varphi_3)\psi_1 - (\psi_2\varphi_3)\psi_2\right].$$

We can now determine β_3 by means of condition 1), provided it is not true that

$$\varphi_3 - (\psi_1\varphi_3)\psi_1 - (\psi_2\varphi_3)\psi_2 \equiv 0. \tag{44}$$

But (44) cannot be satisfied, otherwise φ_1, φ_2, φ_3 would be linearly dependent, which is contrary to hypothesis. Hence β_3 can be so determined that $\psi_3(x)$ satisfies the conditions 1) and 2). This process can be continued until q functions $\psi_\alpha(x)$ are obtained, satisfying conditions 1) and 2), or, as we say, form a *normalized orthogonal system.*

The ψ_α's so obtained are real, they are furthermore linearly independent. For, suppose we had

$$c_1\psi_1(x) + c_2\psi_2(x) + \quad . \quad . \quad . \quad + c_q\psi_q(x) \equiv 0.$$

Multiply by $\psi_\alpha(x)$ and integrate from a to b. On account of (39) and (40) we would obtain $c_\alpha = 0$.

A linear transformation

$$y_i = c_{i1}x_1 + \quad . \quad . \quad . \quad + c_{in}x_n \quad (i = 1, \quad . \quad . \quad . \quad , n)$$

is said to be orthogonal if

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2.$$

The transformation of rectangular coordinates with fixed origin is a transformation of this kind.

The condition on the coefficients which insures this property of the transformation is

$$\sum_{j=1}^{n} c_{ij}c_{kj} = \begin{cases} 1, & i = K \\ 0, & i \neq K. \end{cases} \tag{45}$$

Now consider the definite integral

$$\int_a^b \psi_i(x)\psi_k(x)dx$$

as the limit of a sum

$$\lim_{h \to 0} \sum_j h\psi_i(x_j)\psi_k(x_j) = \int_a^b \psi_i(x)\psi_k(x)dx. \tag{46}$$

Then the analogy between equations (39), (40), and (45) becomes apparent.

c) Complete Normalized Orthogonal System of Fundamental Functions.—To each root of $D(\lambda) = 0$ there belongs such a normalized orthogonal system of fundamental functions. Now we have shown that $D(\lambda) = 0$ has at least one root. There may be a finite or an infinite number of such roots. If they are infinite in number it follows from the theory of permanently convergent power series that they constitute a *denumerable* set and they may be arranged in the order of the magnitude of their absolute values:

$$|\lambda_1| \leqq |\lambda_2| \leqq \quad . \quad . \quad . \quad \leqq |\lambda_n| \leqq |\lambda_{n+1}| \leqq \quad . \quad . \quad .$$

and each λ_i has a definite index q_i. We have then the following table:

Characteristic constant	Index	Normalized orthogonal fundamental functions
λ_1	q_1	$\psi_1^1, \ . \ . \ . \ . \ , \psi_{q_1}^1$
λ_2	q_2	$\psi_1^2, \ . \ . \ . \ . \ , \psi_{q_2}^2$
.	.	. . . . .
.	.	. . . . .
.	.	. . . . .
λ_n	q_n	$\psi_1^n, \ . \ . \ . \ . \ . \ , \psi_{q_n}^n$

We now change the notation and denote the functions ψ in the order in which they stand in the following line:

$$\psi_1^1, \ . \ . \ . \ , \psi_{q_1}^1, \psi_1^2, \ . \ . \ . \ . \ , \psi_{q_2}^2, \ . \ . \ . \ . \ , \psi_1^n, \ . \ . \ . \ , \psi_{q_n}^n$$

by the symbols $\psi_1, \psi_2, \ . \ . \ . \ . \ , \psi_r, \ . \ . \ .$ and the characteristic constants to which they belong by $\lambda_1, \lambda_2, \ . \ . \ . \ .$,

$\lambda_r, \ldots$ where some of the λ's may be equal, for instance $\lambda_1 = \lambda_2 = \ldots = \lambda_{q_1}; \lambda_{q_1+1} = \lambda_{q_1+2} = \ldots = \lambda_{q_1+q_2}$. Such a system of fundamental functions ψ_r is called a complete normalized orthogonal system of fundamental functions. We sum up the preceding discussion in the following theorem:

Theorem VI. *To every real symmetric kernel there belongs a complete normalized orthogonal system of fundamental functions $\psi_r(x)$, with the following properties:*

1) *$\psi_r(x)$ is a fundamental function belonging to λ_r*

$$\psi_r(x) = \lambda_r \int_a^b K(x, t)\psi_r(t)dt.$$

2) $\int_a^b \psi_r^2(x)dx = 1.$

3) $\int_a^b \psi_r(x)\psi_s(x)dx = 0, \quad (r \neq s).$

4) *$\psi_r(x)$ is real.*

5) *Every fundamental function $\varphi(x)$ is expressible in the form*

$$\varphi(x) = c_{r_1}\psi_{r_1}(x) + \ldots + c_{r_m}\psi_{r_m}(x).$$

Proof of 3).—If ψ_r, ψ_s belong to the same λ, they are orthogonal by construction, if to different λ's by Theorem III.

Proof of 5).—$\varphi(x)$ belongs to a certain characteristic constant λ_r and is, therefore, linearly expressible in terms of the fundamental functions $\psi_1^r(x), \ldots, \psi_q^r$, if we use the notation of the table.

Example.—For the problem of the vibrating string we had the kernel:

$$K(x, t) = \begin{cases} (1 - t)x, 0 \leqq x \leqq t \\ (1 - x)t, t \leqq x \leqq 1 \end{cases} \tag{47}$$

with the characteristic constants $\lambda_n = n^2\pi^2$, $n = 1, 2 \ldots$ of index 1, and the corresponding fundamental functions $\varphi_n(x) = \beta \sin n\pi x$. These functions $\varphi_n(x)$ will form a complete normalized orthogonal system of fundamental functions, if

$$\beta = \frac{1}{\sqrt{\int_0^1 \sin^2 n\pi\, x\, dx}} = \pm\sqrt{2}.$$

Our fundamental functions are then

$$\psi_n(x) = \sqrt{2}\sin n\pi x.$$

III. Expansion of an Arbitrary Function According to the Fundamental Functions of a Complete Normalized Orthogonal System

62. a) Problem of the Vibrating String Resumed. The problem was to determine $y(x, t)$ so as to satisfy the following conditions (see §26):

(48) 1) $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$

(49) 2) $y(0, t) = 0, y(1, t) = 0.$

(50) 3) $y(x, 0) = f(x)$, $f(x)$ an arbitrary given function.
$y_t(x, 0) = F(x)$, $F(x)$ an arbitrary given function.

We attempted to find a solution in the form

$$y = \varphi(t)u(x)$$

and found that $u(x)$ must satisfy

1) $\frac{d^2u}{dx^2} + \lambda u = 0$

2) $u(0) = 0, \; u(1) = 0,$

and that $\varphi(t)$ must satisfy the equation

$$\frac{d^2\varphi}{dt^2} + \lambda c^2\varphi = 0.$$

The boundary problem in u had non-trivial solutions only when

$$\lambda = \lambda_n = n^2\pi^2,$$

n a positive integer, the solutions being

$$u = \beta \sin n\pi x.$$

To complete the solution of the partial differential equation (48) with the conditions (49) and (50), we must next integrate the differential equation for $\varphi(t)$ for the values $\lambda = n^2\pi^2$:

$$\frac{d^2\varphi}{dt^2} + n^2\pi^2c^2\varphi = 0.$$

The most general solution of this equation is

$$\varphi(t) = A_n \cos n\pi ct + B_n \sin n\pi ct$$

(A_n, B_n, arbitrary constants), which gives for a solution of the problem of the vibrating string

$$y = (A_n \cos n\pi ct + B_n \sin n\pi ct) \sin n\pi x.$$

This expression for y will satisfy (48) and (49), but, in general, it will not satisfy (50). In order to obtain a solution of (48) which will satisfy both (49) and (50) we notice that, owing to the linear character of (48), the series

$$(51)\qquad y = \sum_{n=1}^{\infty} (A_n \cos n\pi ct + B_n \sin n\pi ct) \sin n\pi x$$

will also satisfy (48) and (49), provided it is convergent and admits of two successive term-by-term differentiations with respect to t and x.

Assuming this condition satisfied, it remains, then, so to determine the constants A_n and B_n that (50) is satisfied:

$$y(x, 0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = f(x)$$

$$y_t(x, 0) = \sum_{n=1}^{\infty} B_n\, n\pi c \sin n\pi x = F(x).$$

But this is equivalent to asking us to develop the arbitrarily given functions $f(x)$ and $F(x)$ into sine series, or, since

$\psi_n(x) = \sqrt{2} \sin n\pi x$, into a series proceeding according to the complete normalized orthogonal system of fundamental functions of the special kernel $K(x, t)$ defined by (47).

b) Determination of the Coefficients in the General Problem. The problem just considered is a particular case of the following more general problem: Given a symmetric kernel $K(x, t)$ with its complete normalized orthogonal system of fundamental functions ψ_ν with corresponding characteristic constants λ_ν, it is possible to expand an arbitrary continuous function $f(x)$ in the form

$$f(x) = \sum_\nu c_\nu \psi_\nu(x).$$

Theorem VII.—*If $f(x)$ is a continuous function and is expressible in the form*

$$f(x) = \sum_\nu c_\nu \psi_\nu(x) \tag{52}$$

and if this series, if infinite, is uniformly convergent on $\left[ab\right]$, then the coefficients c_ν are given by

$$c_n = \int_a^b f(x)\psi_n(x)dx \equiv (f\psi_n)$$

Proof.—Multiply both sides of (52) by $\psi_n(x)$ and integrate with respect to x from a to b:

$$\int_a^b f(x)\psi_n(x)dx = \int_a^b \sum_\nu c_\nu \psi_\nu(x)\psi_n(x)dx$$

$$= \sum_\nu \int_a^b c_\nu \psi_\nu(x)\psi_n(x)dx$$

$$= c_n,$$

since
$$\int_a^b \psi_\nu(x)\psi_n(x)dx = \begin{cases} 1, & \nu = n \\ 0, & \nu \neq n. \end{cases}$$

IV. Expansion of the Kernel According to the Fundamental Functions of a Complete Normalized Orthogonal System

63. a) Determination of the Coefficients.—$K(x, t)$ with t fixed is a function of x continuous on $\left[ab\right]$. Hence, if we assume that $K(x, t)$ can be expanded into a uniformly convergent series on $\left[ab\right]$:

$$K(x, t) = \sum_\nu c_\nu \psi_\nu(x),$$

then, by Theorem VII above, we have

$$c_n = \int_a^b K(x, t)\psi_n(x)dx. \tag{53}$$

But

$$\psi_n(t) = \lambda_n \int_a^b K(t, x)\psi_n(x)dx$$
$$= \lambda_n \int_a^b K(x, t)\psi_n(x)dx,$$

on account of the symmetry of $K(x, t)$. Therefore

$$c_\nu = \frac{\psi_\nu(t)}{\lambda_\nu}$$

and

$$K(x, t) = \sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}.$$

Thus we obtain the following corollary to Theorem VII.

Corollary.—If $K(x, t) = \sum_\nu c_\nu \psi_\nu(x)$ *and this series, if infinite, is uniformly convergent, then*

$$K(x, t) = \sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}. \tag{54}$$

Equation (54) is what Kneser calls the *bilinear formula.*

If we apply this bilinear formula to

$$K(x, t) = \begin{cases} (1 - t)x, & 0 \leqq x \leqq t \\ (1 - x)t, & t \leqq x \leqq 1, \end{cases}$$

we obtain the hypothetical expansion:[1]

$$K(x, t) = \sum_{n=1}^{\infty} \frac{2 \sin n\pi x \sin n\pi t}{n^2\pi^2}.$$

b) The Bilinear Formula for the Case of a Finite Number of Fundamental Functions.—The bilinear formula holds always if the complete normalized orthogonal system contains only a finite number, m, of fundamental functions ψ_ν. The formula to be proved then reads

$$K(x, t) = \sum_{\nu=1}^{m} \frac{\Psi_\nu(x)\Psi_\nu(t)}{\lambda_\nu}. \tag{55}$$

To show that (55) holds, we show that the difference

$$H(x, t) = K(x, t) - \sum_{\nu=1}^{m} \frac{\Psi_\nu(x)\Psi_\nu(t)}{\lambda_\nu} \equiv 0, \text{ on } R. \tag{56}$$

Now $H(x, t)$ is continuous, real, and symmetric. If, in addition $H \not\equiv 0$, then there would exist at least one characteristic constant of $H(x, t)$ considered as a kernel of an integral equation. Hence, if we can show that $H(x, t)$ has no characteristic constant, then it follows that

$$H(x, t) \equiv 0.$$

Suppose that $\varphi(x)$ is a fundamental function of H belonging to a characteristic constant ρ of H, then φ is continuous and $\not\equiv 0$, and

$$\varphi(x) = \rho \int_a^b H(x, t)\varphi(t)dt.$$

[1] This formula is actually true. For a direct proof, see KNESER, "Die Integralgleichungen und ihre Anwendung," §4.

Whence, substituting for $H(x, t)$ its value from (56),

$$(57)\quad \varphi(x) = \rho\int_a^b K(x, t)\varphi(t)dt - \rho\sum_{\nu=1}^{m}\frac{\psi_\nu(x)}{\lambda_\nu}\int_a^b \psi_\nu(t)\varphi(t)dt.$$

Multiply both sides of (57) by $\psi_n(x)$ and integrate with respect to x from a to b, then

$$\begin{aligned}\int_a^b \varphi(x)\psi_n(x)dx &= \rho\int_a^b\int_a^b K(x, t)\psi_n(x)\varphi(t)dtdx \\ &\quad - \rho\sum_{\nu=1}^{m}\frac{1}{\lambda_\nu}\int_a^b \psi_\nu(x)\psi_n(x)dx\int_a^b \psi_\nu(t)\varphi(t)dt \\ &= \frac{\rho}{\lambda_n}\int_a^b \psi_n(t)\varphi(t)dt - \frac{\rho}{\lambda_n}\int_a^b \psi_n(t)\varphi(t)dt \\ &= 0, \qquad (n = 1, \ldots, m).\end{aligned}$$

Therefore, from (57),

$$\varphi(x) = \rho\int_a^b K(x, t)\varphi(t)dt.$$

By hypothesis $\varphi(x)$ is continuous and $\not\equiv 0$, and so this equation shows that $\varphi(x)$ is a fundamental function of $K(x, t)$ belonging to ρ. Therefore

$$\varphi(x) = c_1\psi_1(x) + \cdots + c_m\psi_m(x),$$

whence

$$\begin{aligned}\int_a^b \varphi(x)\psi_n(x)dx &= \int_a^b \psi_n(x)\sum_{\nu=1}^{m}c_\nu\psi_\nu(x)dx \\ &= c_n = 0.\end{aligned}$$

Therefore $\varphi(x) \equiv 0$, which constitutes a contradiction. Therefore $H(x, t)$ has no characteristic constant and $H(x, t) \equiv 0$, and hence the bilinear formula holds for m finite. This gives us the following theorem:

Theorem IX.—*If there are a finite number m of characteristic constants λ_ν, then*

$$K(x, t) = \sum_{\nu=1}^{m} \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}.$$

c) The Bilinear Formula for Kernels Having an Infinite Number of Characteristic Constants.—We proceed to prove the following theorem:

Theorem X.—*If there are an infinite number of characteristic constants and* $\sum_{\nu=1}^{\infty} \frac{\psi_\nu(x)\psi_\nu(t)}{\psi_\nu}$ *is uniformly convergent in x and t on R, then*

$$K(x, t) = \sum_{\nu=1}^{\infty} \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}.$$

Proof.—The proof is entirely analogous to the proof of Theorem IX. Form

$$H(x, t) \equiv K(x, t) - \sum_{\nu=1}^{\infty} \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}. \tag{58}$$

H is continuous, for K is continuous, and the sum of the infinite series in (58) is a continuous function, being a uniformly convergent series the terms of which are continuous.[1] Furthermore, H is real, for K and ψ_ν are real. Finally, H is symmetric, for K is symmetric, and each term of the infinite series in (58) is symmetric. Hence it follows, as under *b*), that if $H(x, t)$ has no characteristic constant, then $H \equiv 0$. Suppose $H(x, t)$ had a characteristic constant ρ. Let $\varphi(x)$ be a fundamental function for H, belonging to the characteristic constant ρ, which implies that $\varphi(x)$ is continuous and $\not\equiv 0$ and satisfies

$$\varphi(x) = \rho \int_a^b H(x, t)\varphi(t)dt,$$

which, on account of (58), may be written

$$\varphi(x) = \rho \int_a^b K(x, t)\varphi(t)dt - \rho \int_a^b \sum_{\nu=1}^{\infty} \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}\varphi(t)dt. \tag{59}$$

[1] GOURSAT-HEDRICK, "Mathematical Analysis," vol. 1, §173.

The series in (59) is again uniformly convergent as to x in $\left[ab\right]$. Multiply both sides of (59) by $\psi_n(x)$ and integrate with respect to x from a to b. We obtain

$$\int_a^b \psi_n(x)\varphi(x)dx = \rho\int_a^b\int_a^b K(x, t)\varphi(t)\psi_n(x)dtdx \\ - \rho\sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu}\int_a^b \psi_\nu(x)\psi_n(x)dx\int_a^b \psi_\nu(t)\varphi(t)dt,$$

which reduces to

$$\int_a^b \psi_n(x)\varphi(x)dx = \frac{\rho}{\lambda_n}\int_a^b \psi_n(t)\varphi(t)dt - \frac{\rho}{\lambda_n}\int_a^b \psi_n(t)\varphi(t)dt \\ = 0,\ (n).$$

Therefore, from (59),

$$\varphi(x) = \rho\int_a^b K(x, t)\varphi(t)dt.$$

Hence we infer, as under b), that $\varphi(x) \equiv 0$, which contradicts our assumption $\varphi(x) \not\equiv 0$. Therefore, $H(x, t)$ has no characteristic constant and, hence, $H(x, t) \equiv 0$.

Since a series with a finite number of terms, functions of x and t, is always uniformly convergent, we may combine Theorems IX and X into the following

Theorem XI.—*If* $\left\{\psi_\nu(x)\right\}$ *are a complete normalized orthogonal system of fundamental functions belonging to the characteristic constant* λ_ν, *for the real symmetric kernel* $K(x, t)$, *and*

$$\frac{1}{\lambda_\nu}\sum_\nu \psi_\nu(x)\psi_\nu(t)$$

is uniformly convergent on R, *then*

$$K(x, t) = \sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}.$$

64. The Complete Normalized Orthogonal System for the Iterated Kernel $K_n(x, t)$.—The series $\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}$ is not always uniformly convergent and therefore the bilinear formula does not hold for every kernel $K(x, t)$. But we shall be able to show that it does always hold for the iterated kernels. For this purpose we prove first the following:

Theorem XII.—*If* $\left\{\psi_\nu(x)\right\}$ *are a complete normalized orthogonal system of fundamental functions for* $K(x, t)$ *belonging to the characteristic constants* $\left\{\lambda_\nu\right\}$, *then* $\left\{\psi_\nu(x)\right\}$ *are a complete normalized orthogonal system of fundamental functions for* $K_n(x, t)$ *belonging to the characteristic constants* $\left\{\lambda_\nu^n\right\}$.

Proof.—*a*) By assumption, $\psi_\nu(x)$ is continuous and $\not\equiv 0$ and

(61) $$\psi_\nu(x) = \lambda_\nu \int_a^b K(x, t)\psi_\nu(t)dt,$$

whence we derive

$$\int_a^b K(z, x)\psi_\nu(x)dx = \lambda_\nu \int_a^b \int_a^b K(z, x)K(x, t)\psi_\nu(t)dtdx,$$

which, from (61) and our definitions of iterated kernels, may be written

$$\frac{\psi_\nu(z)}{\lambda_\nu} = \lambda_\nu \int_a^b K_2(z, t)\psi_\nu(t)dt,$$

or, if we multiply by λ_ν and put x in place of z,

(62) $$\psi_\nu(x) = \lambda_\nu^2 \int_a^b K_2(x, t)\psi_\nu(t)dt.$$

From (62) we derive

$$\psi_\nu(x) = \lambda_\nu^3 \int_a^b K_3(x, t)\psi_\nu(t)dt$$

in the same way that (62) was derived from (61). Continuing this process, it is easy to show by mathematical induction that

$$(63) \qquad \psi_\nu(x) = \lambda_\nu^n \int_a^b K_n(x, t)\psi_\nu(t)dt.$$

Therefore, λ_ν^n is a characteristic constant of $K_n(x, t)$, and $\psi_\nu(x)$ is a fundamental function of $K_n(x, t)$ belonging to λ_ν^n.

b) It is still necessary to show that $K_n(x, t)$ has no other characteristic constants than the λ_ν^n, and that every fundamental function $\varphi(x)$ of $K_n(x, t)$ can be expressed in the form

$$\varphi(x) = C_{\nu_1}\psi_{\nu_1}(x) + \quad . \quad . \quad . \quad + C_{\nu_r}\psi_{\nu_r}(x).$$

Let ρ be any characteristic constant of $K_n(x, t)$ and $\varphi(x)$ be a fundamental function of $K_n(x, t)$ belonging to ρ:

$$(64) \qquad \varphi(x) = \rho \int_a^b K_n(x, t)\varphi(t)dt, \varphi(x) \not\equiv 0.$$

We must then prove that

1) $\rho = \lambda_\nu^n$.

2) $\varphi(x) = \sum_{i=1}^{r} C_{\nu_i}\psi_{\nu_i}(x)$.

Let $h_1, h_2, \quad . \quad . \quad . \quad , h_n$ be the nth roots of ρ:

$$(65) \qquad h_\nu^n = \rho, \ (\nu = 1, \quad . \quad . \quad . \quad , n).$$

Build up the functions $nG_i(x)$ as follows:

$$(66) \quad nG_i(x) = \varphi(x) + h_i \int_a^b K(x, t)\varphi(t)dt + h_i^2 \int_a^b K_2(x, t)\varphi(t)dt + \quad . \quad . \quad . \quad + h_i^{n-1} \int_a^b K_{n-1}(x, t)\varphi(t)dt,$$

for $i = 1, \quad . \quad . \quad . \quad , n$. Add these equations and cancel a factor n. We obtain

$$(67) \qquad G_1(x) + \quad . \quad . \quad . \quad + G_n(x) = \varphi(x),$$

since $h_1^s + h_2^s + \ldots + h_n^s = 0$,

$$(s = 1, \ldots, n-1).$$

Multiply both sides of (66) by $K(z, x)$ and integrate. We obtain

$$n\int_a^b K(z, x)G_i(x)dx = \int_a^b K(z, x)\varphi(x)dx + \sum_{s=1}^{n-1} h_i^s \int_a^b \int_a^b K(z, x)K_s(x, t)\varphi(t)dtdx.$$

Multiply the members of this equation by h_i, put $h_i^n = \rho$ by (65), make use of (64) for $x = z$, $t = x$, and of the definitions of the iterated kernels. We obtain

$$nh_i\int_a^b K(z, x)G_i(x)dx = h_i\int_a^b K(z, t)\varphi(t)dt + h_i^2\int_a^b K_2(z, t)\varphi(t)dt + \ldots + h_i^{n-1}\int_a^b K_{n-1}(z, t)\varphi(t)dt + \varphi(z),$$

which, on account of (66), may be written, after canceling a factor n,

$$h_i\int_a^b K(z, x)G_i(x)dx = G_i(z).$$

Suppose for some value of i that $G_i(x) \not\equiv 0$, then h_i is a characteristic constant of $K(x, t)$ and $G_i(x)$ is a fundamental function of $K(x, t)$ belonging to h_i. Therefore, there exists a value m of ν, such that $h_i = \lambda_m$ and hence $\rho = \lambda_m^n$. Since $G_i(x)$ is a fundamental function of $K(x, t)$, it can be written in the form

(68) $$G_i(x) = C_{i_1}\psi_{i_1} + \ldots + C_{i_k}\psi_{i_k}.$$

But the $G_i(x)$ do not all vanish identically, otherwise from (67) we would have $\varphi(x) \equiv 0$, which cannot be. Hence

$$\rho = \lambda_m^n,$$

and from (67) and (68) we have

$$\varphi(x) = C_{\nu_1}\psi_{\nu_1}(x) + \ . \ . \ . + C_{\nu_r}\psi_{\nu_r},$$

which completes the proof of the theorem.

V. Auxiliary Theorems

65. Bessel's Inequality.—For the further discussion of the bilinear formula we need the following theorem:

Theorem XIII.—*Given*

1) $f(x)$, *real and continuous.*

2) ψ_s, $(s = 1, \ . \ . \ . \ , m)$, *real and continuous, constituting a normalized orthogonal set:*

$$\int_a^b \psi_r\psi_s dx = \begin{cases} 1, & r = s \\ 0, & r \neq s, \end{cases}$$

then

(69) $$\sum_{s=1}^{m}\left[\int_a^b f(x)\psi_s(x)dx\right]^2 \leqq \int_a^b \left[f(x)\right]^2 dx.$$

Proof.—Let C_s be *any* real constants, then

$$\int_a^b \left[f(x) - \sum_{s=1}^{m} C_s\psi_s(x)\right]^2 dx \geqq 0,$$

whence, after squaring the expression in the bracket,

(70) $$\int_a^b \left[f(x)\right]^2 dx - 2\sum_{s=1}^{m} C_s \int_a^b f(x)\psi_s(x)dx + \int_a^b \left[\sum_{s=1}^{m} C_s\psi_s(x)\right]^2 dx \geqq 0.$$

But

$$\int_a^b \left[\sum_{s=1}^{m} C_s\psi_s(x)\right]^2 dx = \sum_{s=1}^{m} C_s^2 \int_a^b \psi_s^2 dx + 2\sum_{(r,s)} C_r C_s \int_a^b \psi_r\psi_s dx.$$

Now

$$\int_a^b \psi_s^2 dx = 1 \text{ and } \int_a^b \psi_r \psi_s dx = 0$$

by hypothesis. Therefore,

$$\int_a^b \left[\sum_{s=1}^m C_s \psi_s(x)\right]^2 dx = \sum_{s=1}^m C_s^2.$$

But C_s are any real constants and so we may choose

$$C_s = \int_a^b f(x)\psi_s(x)dx.$$

For this choice of C_s, equation (70) becomes

$$\int_a^b \left[f(x)\right]^2 dx - 2\sum_{s=1}^m \left(\int_a^b f(x)\psi_s(x)dx\right)^2 + \sum_{s=1}^m \left(\int_a^b f(x)\psi_s(x)dx\right)^2 \geq 0,$$

whence we obtain (69).

We now apply (69) to the particular case where

$$f(x) = K(x, t)$$

for a fixed t, and the functions ψ_s are a normalized orthogonal system of fundamental functions of $K(x, t)$ belonging to λ_s.

Equation (69) now becomes

$$(71) \qquad \sum_{s=1}^m \left(\int_a^b K(x, t)\psi_s(x)dx\right)^2 \leqq \int_a^b \left[K(x, t)\right]^2 dx.$$

But

$$\psi_s(x) = \lambda_s \int_a^b K(x, t)\psi_s(t)dt,$$

whence

$$(72) \qquad \begin{aligned} \psi_s(t) &= \lambda_s \int_a^b K(t, x)\psi_s(x)dx \\ &= \lambda_s \int_a^b K(x, t)\psi_s(x)dx, \end{aligned}$$

on account of the symmetry of $K(x, t)$. Now $K(x, t)$ is continuous on R and, therefore,

(73) $$|K(x, t)| \leq G \quad \text{(a constant)}.$$

Applying (72) and (73) to (71), we obtain

$$\sum_{s=1}^{m}\left(\frac{\psi_s(t)}{\lambda_s}\right)^2 \leq G^2(b-a).$$

Thus we have the following corollary to Theorem XIII:

Corollary.—If $\psi_s(x)$, $(s = 1, \ldots, m)$ *are normalized orthogonal fundamental functions of the kernel* $K(x, t)$, *belonging to the characteristic constant* λ_s, *then*

(74) $$\sum_{s=1}^{m}\left(\frac{\psi_s(x)}{\lambda_s}\right)^2 \leq G^2(b-a)$$

From (74) we obtain by integrating from a to b

$$\sum_{s=1}^{m}\frac{1}{\lambda_s^2}\int_a^b\Big[\psi_s(x)\Big]^2 dx \leq G^2(b-a)^2.$$

But $$\int_a^b\Big[\psi_s(x)\Big]^2 dx = 1 \text{ by hypothesis.}$$

Therefore

(75) $$\sum_{s=1}^{m}\frac{1}{\lambda_s^2} \leq G^2(b-a)^2,$$

the inequality holding for *any* finite number of λ_s and hence for the first m of the λ_s's.

Theorem XIV.—$\sum_{s=1}\frac{1}{\lambda_s^2}$ *is convergent.*

Proof.—The proof follows from (75) by applying the principle of monotony: if u_i are real and positive and

$$u_1 + u_2 + \ldots + u_n \leq A, \ (n)$$

that is bounded, then

$$\sum_{n=1}^{\infty} u_n$$

is convergent.

66. Proof of the Bilinear Formula for the Iterated Kernel $K_n(x, t)$ for $n \leq 4$.—In Theorem XI it was stated that if $\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}$ was uniformly convergent, then

$$\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu} = K(x, t).$$

We have stated previously that this series is not always uniformly convergent and, therefore, the bilinear formula does not hold for every kernel. But we shall be able to prove that it does always hold for the iterated kernels.

Since, according to Theorem XII, $\psi_\nu(x)$ are a complete normalized orthogonal system of fundamental functions for $K_n(x, t)$ belonging to $\lambda_\nu{}^n$, the bilinear formula for the iterated kernel, if true, would read

(76) $$\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu{}^n} = K_n(x, t),\ n > 1.$$

By referring to Theorem XI just quoted, we see that (76) will be proved if we can show that $\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu{}^n}$ is uniformly convergent. We prove this first for the case $n = 4$. We desire then to show the uniform convergence of

(77) $$\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu{}^4} \text{ on } R.$$

We apply the general test for the uniform convergence of the series

$$\sum_{\nu=1}^{\infty} u_\nu(x, t), \text{ on } R.$$

For any $\epsilon > 0$ we can assign an N depending on ϵ, but not on x and t (N_ϵ) such that

$$|u_{n+1}(x, t) + u_{n+2}(x, t) + \ .\ .\ .\ + u_{n+p}(x, t)| < \epsilon,\ (R)$$

for every $n > N_\epsilon$ for any integer $p > 0$.

Use the notation

$$\Delta_{np} = \sum_{\nu=n+1}^{n+p} \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu^4}.$$

We have the inequalities

$$(78) \quad |\Delta_{np}| \leqq \sum_{\nu=n+1}^{n+p} \frac{|\psi_\nu(x)\psi_\nu(t)|}{\lambda_\nu^4} \leqq \sum_\nu \frac{|\psi_\nu(x)||\psi_\nu(t)|}{|\lambda_\nu||\lambda_\nu|} \sum_\nu \frac{1}{\lambda_\nu^2}.$$

We now make use of the inequality

$$(79) \qquad \sum a_\nu b_\nu \leqq \sqrt{\sum a_\nu^2}\sqrt{\sum b_\nu^2}$$

(a_ν, b_ν real and positive), which arises from the following consideration: given the matrix

$$\left\| \begin{matrix} a_1 & a_2 & . & . & . & a_p \\ b_1 & b_2 & . & . & . & b_p \end{matrix} \right\|, \text{ form the product}$$

$$(80) \quad \left\| \begin{matrix} a_1 & a_2 & . & . & . & a_p \\ b_1 & b_2 & . & . & . & b_p \end{matrix} \right\|^2 = \begin{vmatrix} \sum_{\nu=1}^{p} a_\nu^2 & \sum_{\nu=1}^{p} a_\nu b_\nu \\ \sum_{\nu=1}^{p} a_\nu b_\nu & \sum_{\nu=1}^{p} b_\nu^2 \end{vmatrix}$$

$$= \sum_{(\mu,\nu)=1}^{p} \begin{vmatrix} a_\mu & a_\nu \\ b_\mu & b_\nu \end{vmatrix}^2 \geqq 0.$$

From the inequality in (80) follows (79), which is the algebraic analogue of Schwarz's inequality (21) of §57. Now apply (79) to (78) with $a_\nu \equiv \left|\frac{\psi_\nu(x)}{\lambda_\nu}\right|$, $b_\nu \equiv \left|\frac{\psi_\nu(t)}{\lambda_\nu}\right|$. We obtain

$$\sum_{\nu=n+1}^{n+p} \left|\frac{\psi_\nu(x)}{\lambda_\nu}\right| \left|\frac{\psi_\nu(t)}{\lambda_\nu}\right| \leqq \sqrt{\sum_{\nu=n+1}^{n+p} \frac{\psi_\nu^2(x)}{\lambda_\nu^2}} \sqrt{\sum_{\nu=n+1}^{n+p} \frac{\psi_\nu^2(t)}{\lambda_\nu^2}}.$$

But by (74)

$$(81) \quad \sum_{\nu=n+1}^{n+p} \frac{\psi_\nu^2(x)}{\lambda_\nu^2} \leqq G^2(b-a), \sum_{\nu=n+1}^{n+p} \frac{\psi_\nu^2(t)}{\lambda_\nu^2} \leqq G^2(b-a).$$

By Theorem XIV, $\sum \frac{1}{\lambda_\nu^2}$ is convergent, and thus, by the general theorem on convergent series,

$$\lim_{n\to\infty} d_{np} = 0,$$

where

(82) $$d_{np} = \sum_{\nu=n+1}^{n+p} \frac{1}{\lambda_\nu^2}.$$

Substitution from (81) and (82) in (78) gives

$$|\Delta_{np}| \leq G^2(b-a)d_{np}.$$

The right member of this inequality is independent of x and t, therefore (77) is absolutely and uniformly convergent in x and t on R. Thus we have proved the following theorem:

Theorem XV.—$\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu^4}$ *is absolutely and uniformly convergent in x and t on R, and*

$$\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu^4} = K_4(x, t).$$

We can now show that (76) holds for $n > 4$. Suppose that (76) holds in the nth instance:

(83) $$K_n(x, t) = \sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu^n},$$

the series being uniformly convergent in x and t on R. Multiply both sides of (83) by $K(z, x)$ and integrate with respect to x from a to b. We obtain

(84) $$\int_a^b K(z, x)K_n(x, t)dx = \sum_\nu \frac{\psi_\nu(t)}{\lambda_\nu^n}\int_a^b K(z, x)\psi_\nu(x)dx$$

and the series on the right is again uniformly convergent as to x and t. But

$$\int_a^b K(z, x)\psi_\nu(x)dx = \frac{\psi_\nu(z)}{\lambda_\nu}$$

and

$$\int_a^b K(s, x)K_n(x, t)dx = K_{n+1}(s, t).$$

We obtain, then, from (84), if we put $s = x$,

$$K_{n+1}(x, t) = \sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu^{n+1}}.$$

The induction is now complete and (76) holds for $n \geq 4$. Kowalewski[1] gives a proof that (76) holds for $n = 2$, whence from our induction proof, (76) holds for $n = 3$, and therefore for $n > 1$.

67. An Auxiliary Theorem of Schmidt.—Before we can take up the problem of the expansion of an arbitrary function according to the fundamental functions of a symmetric kernel, we need another auxiliary theorem due to E. Schmidt.

Theorem XVI.—*Let* $h(x)$ *be continuous on* $[ab]$, $K(x, t)$ *real and symmetric with the complete normalized orthogonal system of fundamental functions* $\left\{\psi_\nu(x)\right\}$ *belonging to the characteristic constants* $\left\{\lambda_\nu\right\}$, *then*

A) If $\int_a^b K(x, t)h(t)dt = 0$, *uniformly as to* x, *then*

$$\int_a^b \psi_n(x)h(x)dx = 0, \text{ uniformly as to } n.$$

B) If $\int_a^b \psi_n(x)h(x)dx = 0$, *uniformly as to* n, *then*

$$\int_a^b K(x, t)h(t)dt = 0, \text{ uniformly as to } x.$$

Proof.—*A*) Multiply both members of

$$\int_a^b K(x, t)h(t)dt = 0, \ (x)$$

[1] Kowalewski, "Einführung in die Determinanten Theorie," Teubner, p. 533, §200, 1900.

by $\psi_n(x)$ and integrate. We obtain

(85) $$\int_a^b \int_a^b K(x, t)\psi_n(x)h(t)dtdx = 0,$$

which may be written

$$\int_a^b h(t)\left(\int_a^b K(x, t)\psi_n(x)dx\right)dt = 0.$$

But $$\int_a^b K(x, t)\psi_n(x)dx = \frac{\psi_n(t)}{\lambda_n}.$$

Therefore (85) may be written

$$\frac{1}{\lambda_n}\int_a^b \psi_n(t)h(t)dt = 0, \ (n).$$

Therefore $$\int_a^b \psi_n(t)h(t)dt = 0, \ (n).$$

B) Our hypothesis is that

$$\int_a^b \psi_n(x)h(x)dx = 0, \ (n).$$

We have previously shown that

$$K_4(x, t) = \sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu^4}$$

is uniformly convergent on R. Multiply both members of this equality by $h(x)h(t)$ and integrate. We obtain

$$\int_a^b \int_a^b K_4(x, t)h(x)h(t)dxdt = \sum_\nu \frac{1}{\lambda_\nu^4}\int_a^b \int_a^b \psi_\nu(x) \times h(x)\cdot\psi_\nu(t)h(t)dtdx.$$

The right member of this equality may be written

$$\sum_\nu \frac{1}{\lambda_\nu^4}\int_a^b \psi_\nu(x)h(x)dx\cdot\int_a^b \psi_\nu(t)h(t)dt,$$

which, on account of our hypothesis, vanishes. Therefore

(86) $$\int_a^b \int_a^b K_4(x, t)h(x)h(t)dxdt = 0.$$

But $$K_4(x, t) = \int_a^b K_2(x, s)K_2(t, s)ds,$$

since $K_2(t, s) = K_2(s, t)$ from the symmetry of K. Substitute this value of $K_4(x, t)$ in (86). We obtain

$$\int_a^b \int_a^b \int_a^b K_2(x, s)K_2(t, s)h(x)h(t)dsdxdt = 0,$$

which may be written

$$\int_a^b \left[\int_a^b K_2(x, s)h(x)dx \times \int_a^b K_2(t, s)h(t)dt \right] ds = 0,$$

or

$$\int_a^b \left[\int_a^b K_2(x, s)h(x)dx \right]^2 ds = 0.$$

But $\int_a^b K_2(x, s)h(x)ds$ is a real continuous function of s. Therefore

$$\int_a^b K_2(x, s)h(x)dx \equiv 0, \ (s),$$

which, if we change our notation and, in place of s and x, put x and t, and remember that $K_2(x, t) = K_2(t, x)$, becomes

$$\int_a^b K_2(x, t)h(t)dt \equiv 0, \ (x).$$

Multiply both members of this last equation by $h(x)$ and integrate. We obtain

(87) $$\int_a^b \int_a^b K_2(x, t)h(x)h(t)dxdt = 0.$$

But $$K_2(x, t) = \int_a^b K(x, s)K(t, s)ds.$$

Treat (87) in the same manner as we have just done with (86) and we will finally obtain

$$\int_a^b K(x, t)h(t)dt \equiv 0, \ (x).$$

This completes the proof of Schmidt's auxiliary theorem.

VI. Expansion of an Arbitrary Function According to the Complete Normalized Orthogonal System of Fundamental Functions of a Symmetric Kernel

68. We are now in a position to solve the fundamental problem stated in §62: Given an arbitrary function $f(x)$, to represent it, if possible, in the form

$$f(x) = \sum_{\nu} C_\nu \psi_\nu(x),$$

when the $\psi_\nu(x)$ are the fundamental functions of a complete normalized orthogonal system of the symmetric kernel $K(x, t)$.

We have seen in §62, that if such an expansion is possible and, moreover, uniformly convergent, then the coefficients C_ν must have the values

$$C_\nu = \int_a^b f(x)\psi_\nu(x)dx \equiv (f\psi_\nu),$$

that is

$$f(x) = \sum_{\nu} (f\psi_\nu)\psi_\nu(x).$$

We now prove the following theorem:

Theorem XVII.—*If $f(x)$ can be represented in the form*

(88) $$f(x) = \int_a^b K(x, t)g(t)dt,$$

where $g(x)$ is a function continuous on $\left[ab\right]$, then

$$f(x) = \sum_{\nu} (f\psi_\nu)\psi_\nu(x) \text{ on } \left[ab\right]$$

and the series is uniformly and absolutely convergent on $\left[ab\right]$.

Proof.—a) Convergence Proof. Use the notation

$$\Delta_{np} = \sum_{\nu=n+1}^{n+p} (f\psi_\nu)\psi_\nu(x),\ p > 0.$$

Then, from the general convergence principle stated in §66, the series

$$\sum_{\nu=1}^{\infty}(f\psi_\nu)\psi_\nu(x) \tag{89}$$

converges uniformly on $\left[ab\right]$ if for every $\epsilon > 0$ there exists an N dependent upon ϵ, but not on $x(N_\epsilon)$, such that for every $n > N_\epsilon$ it is true that $|\Delta_{np}| < \epsilon$ on $\left[ab\right]$. Now

$$\begin{aligned}(f\psi_\nu) &= \int_a^b f(x)\psi_\nu(x)dx \\ &= \int_a^b\int_a^b K(x, t)g(t)\psi_\nu(x)dt\,dx \end{aligned} \tag{90}$$

on account of (88).

The right member of (90) may be written

$$\int_a^b g(t)\left(\int_a^b K(x, t)\psi_\nu(x)dx\right)dt.$$

But

$$\int_a^b K(x, t)\psi_\nu(x)dx = \frac{\psi_\nu(t)}{\lambda_\nu}.$$

Therefore (90) becomes

$$(f\psi_\nu) = \frac{1}{\lambda_\nu}(g\psi_\nu). \tag{91}$$

Now

$$|\Delta_{np}| \leqq \sum_{\nu=n+1}^{n+p}|(f\psi_\nu)||\psi_\nu(x)|.$$

Therefore, by (91),

$$|\Delta_{np}| \leq \sum_{\nu=n+1}^{n+p}|(g\psi_\nu)|\cdot\left|\frac{\psi_\nu(x)}{\lambda_\nu}\right|. \tag{92}$$

But from the algebraic analogue (79) of Schwarz's inequality (21), proved in §57, we have

$$\sum_\nu|(g\psi_\nu)|\left|\frac{\psi_\nu(x)}{\lambda_\nu}\right| \leqq \sqrt{\sum_\nu(g\psi_\nu)^2}\sqrt{\sum_\nu\frac{\psi_\nu^{\,2}(x)}{\lambda_\nu^{\,2}}},$$

and from (74) in the corollary to Bessel's inequality

$$\sum_{\nu}\frac{\psi_\nu^2(x)}{\lambda_\nu^2} \leq G^2(b-a),$$

G being the maximum of $|K(x, t)|$ on R.

If, now, we use the notation

$$d_{np} = \sum_{\nu=n+1}^{n+p}(g\psi_\nu)^2$$

we may write

(93) $$|\Delta_{np}| \leq G\sqrt{(b-a)d_{np}}.$$

But

(94) $$\sum_{\nu=1}^{\infty}(g\psi_\nu)^2$$

is convergent, for

$$\sum_{\nu=1}^{n}(g\psi_\nu)^2 = \sum_{\nu=1}^{n}\left(\int_a^b g(x)\psi_\nu(x)dx\right)^2 \leq \int_a^b\left[g(x)\right]^2dx,$$

from Bessel's inequality (69), §65, and $\int_a^b\left[g(x)\right]^2dx$ is a finite fixed constant, independent of n, and therefore

$$\sum_{\nu=1}^{n}(g\psi_\nu)^2$$

is bounded for every n, and hence, by the monotony principle (94), is convergent. Therefore, $\lim\limits_{n\to\infty} d_{np} = 0$. Hence, from (93), $\lim\limits_{n\to\infty}|\Delta_{np}| = 0$, and therefore (89) is uniformly convergent on $\left[ab\right]$, and from (92) we see that it is also absolutely convergent. This completes the first part of the proof. It remains to determine the sum.

b) Summation.—Put

$$h(x) \equiv f(x) - \sum_{\nu}(f\psi_\nu)\psi_\nu(x).$$

From (88), $f(x)$ is continuous, since $g(x)$ is supposed to be continuous and the ψ_ν's are continuous and the series is uniformly convergent, hence h is continuous. We will have proved that the series represents $f(x)$, if we can show

that $$\int_a^b \left[h(x) \right]^2 dx = 0,$$

for then

$$h(x) \equiv 0.$$

Now

$$\int_a^b \left[h(x) \right]^2 dx = (fh) - \sum_\nu (f\psi_\nu)(h\psi_\nu). \tag{95}$$

We compute

$$(h\psi_n) = (f\psi_n) - \sum_\nu (f\psi_\nu)(\psi_\nu\psi_n).$$

But $$(\psi_\nu\psi_n) = \begin{cases} 1, & \nu = n \\ 0, & \nu \neq n. \end{cases}$$

Therefore

$$(h\psi_n) = (f\psi_n) - (f\psi_n) = 0. \tag{96}$$

Whence, from (95),

$$\int_a^b \left[h(x) \right]^2 dx = (fh). \tag{97}$$

But $$(fh) = \int_a^b f(x)h(x)dx$$

and, by hypothesis, as given by (88),

$$f(x) = \int_a^b K(x, t)g(t)dt.$$

Therefore,

$$(fh) = \int_a^b \int_a^b K(x, t)g(t)h(x)dtdx$$

$$= \int_a^b g(t)\left(\int_a^b K(x, t)h(x)dx\right)dt. \tag{98}$$

Now $$\int_a^b \psi_n(x)h(x)dx = 0, (n) \text{ by } (96),$$

hence, by Theorem XVI, §67,

$$\int_a^b K(x, t)h(t)dt = 0, \ (x)$$

and, therefore,

$$\int_a^b K(x, t)h(x)dx = 0, \ (t)$$

whence, from (98),

$$(fh) \equiv 0, \ (x)$$

and, therefore, from (97),

$$h(x) \equiv 0, \ (x),$$

which completes the proof of the theorem. We have the following corollaries to Theorem XVII.

Corollary I.—$(f\psi_\nu) = \dfrac{(g\psi_\nu)}{\lambda_\nu}$ by (91).

Corollary II.—*If $g(x)$ is continuous, then*

(99) $$\int_a^b K(x, t)g(t)dt = \sum_{\nu=1}^{\infty} \frac{(g\psi_\nu)}{\lambda_\nu}\psi_\nu(x)$$

and the series is uniformly and absolutely convergent. This follows from Theorem XVII and Corollary I.

Corollary III.—*If $g(x)$ and $h(x)$ are continuous, then*

$$\int_a^b\int_a^b K(x, t)g(t)h(x)dtdx = \sum_{\nu=1}^{\infty} \frac{(g\psi_\nu)(h\psi_\nu)}{\lambda_\nu}.$$

This is sometimes called *Hilbert's* formula. It follows from (99), if we multiply by $h(x)$ and integrate.

VII. Solution of the Integral Equation

69. Schmidt's Solution of the Non-homogeneous Integral Equation When λ Is Not a Characteristic Constant.—As an application of the last theorem, we will now give Schmidt's solution of the equation

(100) $$f(x) = u(x) - \lambda\int_a^b K(x, t)u(t)dt.$$

A) Necessary Form of the Solution.—Let $u(x)$ be a continuous solution of (100), then

$$(101) \qquad \mathrm{v}(x) \equiv \frac{u(x) - f(x)}{\lambda} = \int_a^b K(x, t)u(t)dt.$$

The function $\mathrm{v}(x)$ satisfies the condition (88) of Theorem XVII and, therefore,

$$\mathrm{v}(x) = \sum_\nu (\mathrm{v}\psi_\nu)\psi_\nu(x),$$

the series being absolutely and uniformly convergent on $\left[ab\right]$. On account of (101) and Corollary I §68, we have

$$(102) \qquad \mathrm{v}(x) = \sum_\nu \frac{(u\psi_\nu)}{\lambda_\nu}\psi_\nu(x).$$

Multiply (100) by $\psi_\nu(x)$ and integrate. We obtain

$$\begin{aligned}(f\psi_\nu) &= (u\psi_\nu) - \lambda\int_a^b\int_a^b K(x, t)u(t)\psi_\nu(x)dtdx \\ &= (u\psi_\nu) - \lambda\int_a^b \frac{\psi_\nu(t)u(t)}{\lambda_\nu}dt \\ &= (u\psi_\nu) - \frac{\lambda}{\lambda_\nu}(u\psi_\nu),\end{aligned}$$

which may be written

$$(\lambda_\nu - \lambda)(u\psi_\nu) = \lambda_\nu(f\psi_\nu).$$

When $\lambda \neq \lambda_\nu$, we have

$$(103) \qquad (u\psi_\nu) = \frac{\lambda_\nu(f\psi_\nu)}{\lambda_\nu - \lambda}.$$

If λ is not a characteristic constant, then $\lambda \neq \lambda_1, \lambda_2, \ldots$, and for such values of λ (103) holds for all values of ν and hence, from (102),

$$\mathrm{v}(x) = \sum_{\nu=1}^{\infty} \frac{(f\psi_\nu)}{\lambda_\nu - \lambda}\psi_\nu(x).$$

Whence from (100) and (101)

$$u(x) = f(x) + \lambda \sum_{\nu=1}^{\infty} \frac{(f\psi_\nu)}{\lambda_\nu - \lambda} \psi_\nu(x). \tag{104}$$

Hence, if (100) has a continuous solution $u(x)$, this solution is unique and is given by (104). Replacing the symbol $(f\psi_\nu)$ by its explicit expression, we have for the solution of (100)

$$u(x) = f(x) + \lambda \sum_{\nu=1}^{\infty} \left[\frac{1}{\lambda_\nu - \lambda} \int_a^b f(t)\psi_\nu(t)dt \right] \psi_\nu(x). \tag{105}$$

If for a given value of λ, the series

$$\sum_{\nu=1}^{\infty} \frac{\psi_\nu(t)\psi_\nu(x)}{\lambda_\nu - \lambda}$$

is uniformly convergent in t, we can write the solution in a form previously obtained:

$$u(x) = f(x) + \lambda \int_a^b K(x, t; \lambda) f(t) dt$$

where

$$K(x, t; \lambda) = \sum_{\nu=1}^{\infty} \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu - \lambda}.$$

The solution given by (105) has an advantage over that given by Fredholm, in that it shows the meromorphic character of the solution with respect to λ and indicates the principal part of $u(x)$ with respect to each pole λ.

B) Sufficiency Proof.—It remains to show that (104) is absolutely and uniformly convergent and satisfies (100).

1) *Uniform and Absolute Convergence.*—The series in (104) may be written as follows

$$\sum_{\nu=1}^{\infty} \frac{1}{1 - \frac{\lambda}{\lambda_\nu}} \frac{(f\psi_\nu)}{\lambda_\nu} \psi_\nu(x). \tag{106}$$

But by Corollary II to Theorem XVII

$$\sum_\nu \frac{(f\psi_\nu)}{\lambda_\nu} \psi_\nu(x)$$

is absolutely and uniformly convergent; and if there is an infinitude of characteristic constants λ_ν, then as $\nu \to \infty$ also $\lambda_\nu \to \infty$, and therefore $\dfrac{1}{1 - \dfrac{\lambda}{\lambda_\nu}} \to 1$, whence also (106) is absolutely and uniformly convergent.

2) (104) *Satisfies* (100).—From (104) we obtain

$$u(x) - \lambda \int_a^b K(x, t)u(t)dt = f(x) + \lambda \sum_\nu \frac{(f\psi_\nu)}{\lambda_\nu - \lambda} \psi_\nu(x)$$
$$- \lambda \int_a^b K(x, t) \left\{ f(t) + \lambda \sum_\nu \frac{(f\psi_\nu)}{\lambda_\nu - \lambda} \psi_\nu(t) \right\} dt.$$

The right-hand member may be written

$$(107) \quad f(x) + \lambda \sum_\nu \frac{(f\psi_\nu)}{\lambda_\nu - \lambda} \psi_\nu(x) - \lambda \int_a^b K(x, t)f(t)dt$$
$$- \lambda^2 \sum_\nu \frac{(f\psi_\nu)}{\lambda_\nu - \lambda} \int_a^b K(x, t)\psi_\nu(t)dt.$$

But $$\int_a^b K(x, t)\psi_\nu(t)dt = \frac{\psi_\nu(x)}{\lambda_\nu},$$

therefore (107) may be written

$$f(x) + \lambda \sum_\nu \frac{(f\psi_\nu)}{\lambda_\nu - \lambda} \psi_\nu(x) \left\{ 1 - \frac{\lambda}{\lambda_\nu} \right\} - \lambda \int_a^b K(x, t)f(t)dt$$

or

$$f(x) + \lambda \sum_\nu \frac{(f\psi_\nu)}{\lambda_\nu} \psi_\nu(x) - \lambda \int_a^b K(x, t)f(t)dt.$$

By (99) this last expression reduces to $f(x)$ and hence

$$u(x) - \lambda \int_a^b K(x, t)u(t)dt = f(x).$$

We have thus proved the following theorem:

Theorem XVIII.—*If $f(x)$ is continuous and λ is not a characteristic constant, then*

$$u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$$

has one and only one continuous solution $u(x)$ given by

$$u(x) = f(x) + \lambda \sum_\nu \left[\frac{1}{\lambda_\nu - \lambda} \int_a^b f(t)\psi_\nu(t)dt \right] \psi_\nu(x),$$

the series being absolutely and uniformly convergent.

70. Schmidt's Solution of the Non-homogeneous Integral Equation When λ Is a Characteristic Constant.—Let us suppose that λ *is* a characteristic constant, for example, $\lambda = \lambda_1$ of index q, so that

$$\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_q.$$

A) As before, we find that

$$(\lambda_\nu - \lambda)(u\psi_\nu) = \lambda_\nu(f\psi_\nu).$$

Now, since $\lambda_\nu = \lambda$, $\nu = 1, 2, \ldots, q$, we have

$$(108) \qquad (f\psi_\nu) = 0, \quad \nu = 1, 2, \ldots, q.$$

We have thus q *necessary* conditions on f. If $\nu > q$, then $\lambda_\nu - \lambda \neq 0$ and

$$(u\psi_\nu) = \frac{\lambda_\nu(f\psi_\nu)}{\lambda_\nu - \lambda} \text{ as before.}$$

Now $\mathrm{v}(x)$ is expressible as before by the formula (102), but for $\nu = 1, \ldots, q$ the constant coefficients $\dfrac{(u\psi_\nu)}{\lambda_\nu}$ are as yet undetermined. So we write

$$\mathrm{v}(x) = C_1\psi_1(x) + \cdots + C_q\psi_q(x) + \sum_{\nu=q+1}^{\infty} \frac{(f\psi_\nu)}{\lambda_\nu - \lambda}\psi_\nu(x)$$

whence

(109) $$u(x) = f(x) + C_1\psi_1(x) + \cdots + C_q\psi_q(x) + \lambda\sum_{\nu=q+1}^{\infty}\frac{(f\psi_\nu)}{\lambda_\nu - \lambda}\psi_\nu(x).$$

B) That (109) converges is seen as before. To show that (109) satisfies (100) we first show that

$$u_0(x) = f(x) + \lambda\sum_{\nu=q+1}^{\infty}\frac{(f\psi_\nu)}{\lambda_\nu - \lambda}\psi_\nu(x)$$

satisfies (100). For this proceed as before and we find, after some reductions,

$$u_0(x) - \lambda\int_a^b K(x, t)u_0(t)dt = f(x) + \lambda\sum_{\nu=q+1}^{\infty}\frac{(f\psi_\nu)}{\lambda_\nu}\psi_\nu(x) - \lambda\sum_{\nu=1}^{\infty}\frac{(f\psi_\nu)}{\lambda_\nu}\psi_\nu(x).$$

But $(f\psi_\nu) = 0$ for $\nu = 1, 2, \ldots, q$, by (108) and thus we get

(110) $$u_0(x) - \lambda\int_a^b K(x, t)u_0(t)dt = f(x).$$

It remains to show that

$$u(x) = u_0(x) + C_1\psi_1(x) + \cdots + C_q\psi_q(x) \equiv u_0(x) + \varphi(x)$$

satisfies (100). Now

$$u(x) - \lambda\int_a^b K(x, t)u(t)dt = u_0(x) - \lambda\int_a^b K(x, t)u_0(t)dt + \varphi(x) - \lambda\int_a^b K(x, t)\varphi(t)dt = f(x) \text{ by (110)},$$

for

$$\varphi(x) - \lambda\int_a^b K(x, t)\varphi(t)dt = 0,$$

since

$$\psi_\alpha(x) - \lambda\int_a^b K(x, t)\psi_\alpha(t)dt = 0 \ (\alpha = 1, \ldots, q).$$

We have thus proved the following theorem:

Theorem XIX.—*If $\lambda = \lambda_1$ is a characteristic constant of index q, then*

$$u(x) = f(x) + \lambda_1 \int_a^b K(x, t)u(t)dt$$

has in general no solution. Solutions exist only if

$$(f\psi_\alpha) = 0, \ (\alpha = 1, \ . \ . \ . \ , q).$$

Then there are q^∞ solutions given by

$$(111) \quad u(x) = f(x) + \lambda \sum_{\nu=q+1}^{\infty} \left[\frac{1}{\lambda_\nu - \lambda} \int_a^b f(t)\psi_\nu(t)dt \right] \psi_\nu(x) + C_1\psi_1(x) + \ . \ . \ . \ + C_q\psi_q(x),$$

where $C_1, \ . \ . \ . \ , C_q$ are arbitrary constants.

71. Remarks on Obtaining a Solution.—To obtain the solution of any numerical problem, one must compute $D(\lambda)$ in order to obtain λ_ν. To obtain the ψ_ν one must solve the corresponding homogeneous equation. Having obtained the ψ_ν and the λ_ν, one writes down the solution at once by direct substitution in the proper formula, (104) or (111).

In general, $D(\lambda)$ is an infinite series given by (5) in Chap. III. However, for many simple kernels all but a few of the first terms vanish identically and we obtain $D(\lambda)$ as a polynomial in λ. By the method of §25, Chap. III, $D(\lambda)$ can be obtained in finite form as a polynomial in λ for every kernel $K(x, t)$ which is a polynomial in x and t. The ψ_ν can be computed from the formula for $\varphi_\alpha(x)$ given in Theorem VIII, §21, Chap. III. By the method of §25, Chap. III, ψ_ν can be obtained in finite form for every kernel $K(x, t)$ which is a polynomial in x and t.

EXERCISES

Compute $D(\lambda)$ and obtain the characteristic constants λ_ν for the following symmetric kernels for the specified interval $[ab]$.

Ans.

1. $K(x, t) = 1$, [01]. $D(\lambda) = 1 - \lambda.$
2. $K(x, t) = -1$, [01]. $D(\lambda) = 1 + \lambda.$
3. $K(x, t) = xt$, [01]. $D(\lambda) = 1 - \frac{\lambda}{3}.$
4. $K(x, t) = \sin x \sin t$ $[0, 2\pi]$. $D(\lambda) = 1 - \pi\lambda.$
5. $K(x, t) = e^x e^t [0, \log_e 2]$. $D(\lambda) = 1 - \frac{3}{2}\lambda.$
6. $K(x, t) = x + t$, [01]. $D(\lambda) = 1 - \lambda - \frac{\lambda^2}{12}.$
7. $K(x, t) = x^2 + t^2$, [01]. $D(\lambda) = 1 - \frac{2}{3}\lambda - \frac{4}{45}\lambda^2.$
8. $K(x, t) = x^2 t + x t^2$, [01]. $D(\lambda) = 1 - \frac{\lambda}{2} - \frac{1}{240}\lambda^2.$
9. $K(x, t) = x^2 + xt + t^2$, [01]. $D(\lambda) = 1 - \lambda - \frac{7}{60}\lambda^2 + \frac{1}{2{,}160}\lambda^3.$

Solve the following equations:

10. $u(x) = \int_0^1 u(t)dt.$ $u(x) = C.$

11. $u(x) = x + \lambda \int_0^1 u(t)dt$, $(\lambda \neq 1)$. $u(x) = x + \frac{\lambda}{2(1 - \lambda)}.$

12. $u(x) = \frac{1}{2} - x + \int_0^1 u(t)dt.$ $u(x) = \frac{1}{2} - x + C.$

13. $u(x) = \lambda \int_0^1 (x + t)u(t)dt$ for $\lambda = -6 \pm 4\sqrt{3}$.

Ans. $u_1(x) = C_1(1 + \sqrt{3}x)$ for $\lambda = \lambda_1 = -6 + 4\sqrt{3}.$
$u_2(x) = C_2(1 - \sqrt{3}x)$ for $\lambda = \lambda_2 = -6 - 4\sqrt{3}.$

14. $u(x) = x + \lambda \int_0^1 (x + t)u(t)dt$, $\lambda \neq \lambda_1, \lambda_2$.

$$\textit{Ans. } u(x) = x - \frac{\frac{\lambda}{2}x - 2 - \sqrt{3}}{\lambda + 6 + 4\sqrt{3}} - \frac{\frac{\lambda}{2}x + 2 - \sqrt{3}}{\lambda + 6 - 4\sqrt{3}}$$

$$\text{or } u(x) = \frac{(6\lambda - 12)x - 4\lambda}{\lambda^2 + 12\lambda - 12}.$$

15. $u(x) = (1 - \sqrt{3}x) + (-6 + 4\sqrt{3}) \int_0^1 (x + t)u(t)dt.$

$$\textit{Ans. } u(x) = (1 - \sqrt{3}x) + B(1 + \sqrt{3}x) - \left(1 + \frac{3}{2}x\right).$$

16. $u(x) = (1 + \sqrt{3}x) + (-6 - 4\sqrt{3}) \int_0^1 (x + t)u(t)dt.$

$$\textit{Ans. } u(x) = (1 + \sqrt{3}x) + B(1 - \sqrt{3}x) - \left(1 + \frac{3}{2}x\right).$$

CHAPTER VI

APPLICATIONS OF THE HILBERT-SCHMIDT THEORY

I. Boundary Problems for Ordinary Linear Differential Equations

72. Introductory Remarks.—*a) Formulation of the Problem.*—We have previously considered the boundary problem

$$\frac{d^2u}{dx^2} + \lambda u = 0,\ u(0) = 0,\ u(1) = 0.$$

We shall now discuss the general type of boundary problem of which this is a special case, namely, we take the general homogeneous linear differential equation of the second order with a parameter λ contained linearly in the coefficients of u:

$$P(x)\frac{d^2u}{dx^2} + Q(x)\frac{du}{dx} + \left[R(x) + \lambda S(x)\right]u = 0.$$

We first reduce this equation to a self-adjoint linear differential equation by multiplying it by $\frac{1}{P(x)} e^{\int \frac{Q(x)}{P(x)}dx}$

Putting $p = e^{\int \frac{Q}{P}dx}$, we obtain

$$p\left(u'' + \frac{Q}{P}u' + \frac{R + \lambda S}{P}u\right) = 0,$$

which may be written

(1) $$pu'' + p'u' + (q + \lambda g)u = 0$$

or $$\frac{d}{dx}\left(p\frac{du}{dx}\right) + (q + \lambda g)u = 0.$$

If we make use of the permanent notation

$$L(u) = \frac{d}{dx}\left(p\frac{du}{dx}\right) + qu, \tag{2}$$

equation (1) may be written in the form

$$L(u) + \lambda g(u) = 0. \tag{3}$$

We use also more general boundary conditions, and for the general interval $\left[ab\right]$:

$$\begin{aligned} R_0(u) &\equiv Au(a) + Bu'(a) = 0 \\ R_1(u) &\equiv Cu(b) + Du'(b) = 0, \end{aligned} \tag{4}$$

where A, B, C, D are given constants, of which A and B are not both zero, and C and D are not both zero.

Our problem then is to determine all solutions of class C'' of the differential equation (3) which satisfy the boundary conditions (4).

For this generalized boundary problem we make the hypotheses:

(H_1) *p is of class C', $p \neq 0$ on $[ab]$*

q and g are of class C on $[ab]$.

Then (1) has two linearly independent solutions of class C'' on $\left[ab\right]$: u_1 and u_2. Any other solution of (1) of class C'' is linearly expressible in terms of u_1 and u_2:

$$u = C_1u_1 + C_2u_2.$$

The boundary problem has the trivial solution $u \equiv 0$. Every value of λ for which our boundary problem has a non-trivial solution is called a characteristic constant and those solutions of class C'', non-trivial, which exist for these values of λ are called fundamental functions of the boundary problem.

To the hypothesis (H_1) we add for the present a second hypothesis (H_2) which will be dropped later:

(H_2) $\lambda = 0$ *is not a characteristic constant.*

That is, $L(u) = 0,\ R_0(u) = 0,\ R_1(u) = 0$

has no solution of class C'' other than $u \equiv 0$.

b) Green's Formula.—From our hypothesis (H_1) we obtain the following theorem:

Theorem I.—*If $u(x)$ and $\mathrm{v}(x)$ are any two functions of x of class C'', then*

(5) $$uL(\mathrm{v}) - \mathrm{v}L(u) = \frac{d}{dx}p(u\mathrm{v}' - \mathrm{v}u').$$

Proof.—Given two functions $u(x)$, $\mathrm{v}(x)$ of class C'', form

$$uL(\mathrm{v}) - \mathrm{v}L(u) = \begin{vmatrix} u & pu'' + p'u' + qu \\ \mathrm{v} & p\mathrm{v}'' + p'\mathrm{v}' + q\mathrm{v} \end{vmatrix}$$
$$= \begin{vmatrix} u & pu'' + p'u' \\ \mathrm{v} & p\mathrm{v}'' + p'\mathrm{v}' \end{vmatrix}$$
$$= p(u\mathrm{v}'' - \mathrm{v}u'') + p'(u\mathrm{v}' - \mathrm{v}u')$$
$$= \frac{d}{dx}p(u\mathrm{v}' - \mathrm{v}u').$$

Corollary.—*If u and v are of class C'' and $L(u) = 0$,* $L(\mathrm{v}) = 0$, then

(6) $$p(u\mathrm{v}' - \mathrm{v}u') = \text{constant} \neq 0,$$

if u and v are independent.

For if $C = 0$, then $u\mathrm{v}' - \mathrm{v}u' \equiv 0$ and

$$\frac{u\mathrm{v}' - \mathrm{v}u'}{u^2} \equiv 0.$$

Whence $$\frac{d}{dx}\left(\frac{\mathrm{v}}{u}\right) \equiv 0,$$

and, therefore, $\mathrm{v} = Cu$ and v and u would be linearly dependent, contrary to hypothesis.

c) Consequences of Hypothesis (H_2).—We now establish the following lemmas:

Lemma I.—If u_1, u_2 *are of class* C'' *and* $L(u_1) = 0$, $L(u_2) = 0$, *and* u_1, u_2 *are linearly independent, then*

$R_0(u_1)$, $R_0(u_2)$ *are not both zero and also*

$R_1(u_1)$, $R_1(u_2)$ *are not both zero.*

Proof.—The general solution of class C'' of $L(u) = 0$ is

$$u = c_1u_1 + c_2u_2.$$

Now
$$R_0(u) = c_1R_0(u_1) + c_2R_0(u_2).$$

Therefore, if the lemma is not true, then

$$R_0(u) = 0 \text{ for every } c_1, c_2.$$

But
$$R_1(u) = c_1R_1(u_1) + c_2R_1(u_2),$$

and, if c_1, c_2 are properly chosen, we can make $R_1(u) = 0$. But then we would have

$$L(u) = 0,\ R_0(u) = 0,\ R_1(u) = 0,$$

which is contrary to hypothesis (H_2). Therefore not both $R_0(u_1)$, $R_0(u_2)$ can vanish at once. Likewise, we show that $R_1(u_1)$, $R_1(u_2)$ cannot both vanish simultaneously.

Lemma II.—There exist functions u *and* v *of class* C'', *determined except for a constant factor, such that* $L(u) = 0$, $R_0(u) = 0$, $L(\mathrm{v}) = 0$, $R_1(\mathrm{v}) = 0$. u and v *are linearly independent and the constant factors can be determined so that* $p(u\mathrm{v}' - \mathrm{v}u') = -1$.

Proof.—Let u_1, u_2 be two linearly independent solutions of $L(u) = 0$. Let $u = c_1u_1 + c_2u_2$ and determine c_1 and c_2 so that $R_0(u) = c_1R_0(u_1) + c_2R_0(u_2) = 0$. Now $R_0(u_1)$, $R_0(u_2)$ cannot both be zero. Therefore

$$c_1 = \rho_0R_0(u_2) \text{ and } c_2 = -\rho_0R_0(u_1),$$

and hence

$$u = \rho_o\Big[u_1R_0(u_2) - u_2R_0(u_1)\Big].$$

This is the most general expression for a function u for which $L(u) = 0$, $R_0(u) = 0$.

In like manner, we show that the most general expression for a function v for which

$$L(\mathrm{v}) = 0,\ R_1(\mathrm{v}) = 0$$

is

$$\mathrm{v} = \rho_1\Big[\mathrm{v}_1 R_1(\mathrm{v}_2) - \mathrm{v}_2 R_1(\mathrm{v}_1)\Big].$$

Suppose u and v were not independent, then

$$\mathrm{v} = Cu,\ C \neq 0,$$

whence

$$R_1(u) = \frac{1}{C} R_1(\mathrm{v}) = 0.$$

But by hypothesis $R_0(u) = 0$. This is in contradiction to our hypothesis (H_2).

According to Green's formula

$$p(u\mathrm{v}' - \mathrm{v}u') = C \neq 0;$$

hence the constant factors ρ_0, ρ_1 can be determined so that

$$p(u\mathrm{v}' - \mathrm{v}u') = -1.$$

73. Construction of Green's Function.—Let us take between a and b a fixed value t: $a < t < b$. Then the following theorem holds:

Theorem II.—*For our boundary problem there exists under the assumptions (H_1) and (H_2) one and but one function $K(x, t)$, which, as a function of x, has the following properties:*

A) K is continuous on $\Big[ab\Big]$.

B) K is of class C'' on each of its subintervals $\Big[at\Big]\Big[tb\Big]$, *and $L(K) = 0$ on each subinterval.*

C) $R_0(K) = 0,\ R_1(K) = 0.$

D) $K'(x, t - 0) - K'(x, t + 0) = \dfrac{1}{p(t)}$,

where
$$K'(x, t) = \frac{\partial}{\partial x} K(x, t).$$

This function $K(x, t)$ is called *Green's function*, belonging to the boundary problem.

Proof. If $K(x, t)$ is to satisfy $L(K) = 0$, it must be linearly expressible in terms of the two functions u and v the existence of which have been shown in Lemma II of §72:

$$K(x, t) = \begin{cases} A_0 u + B_0 \mathrm{v}, & [at] \\ A_1 u + B_1 \mathrm{v}, & [tb]. \end{cases}$$

Now demand that $K(x, t)$ satisfy C):

$$R_0(K) = R_0(A_0 u + B_0 \mathrm{v}) = A_0 R_0(u) + B_0 R_0(\mathrm{v}) = 0.$$
$$R_1(K) = R_1(A_1 u + B_1 \mathrm{v}) = A_1 R_1(u) + B_1 R_1(\mathrm{v}) = 0.$$

But $R_0(u) = 0$, and not both $R_0(u)$, $R_0(\mathrm{v})$ can vanish simultaneously, whence $B_0 = 0$. Also, $R_1(\mathrm{v}) = 0$, and not both $R_1(u)$, $R_1(\mathrm{v})$ can vanish simultaneously, whence $A_1 = 0$. Therefore

$$K(x, t) = \begin{cases} A_0 u, & [at] \\ B_1 \mathrm{v}, & [tb]. \end{cases}$$

We now impose the condition A):

$$A_0 u(t) = B_1 \mathrm{v}(t),$$

whence

$$A_0 = \rho \mathrm{v}(t), \; B_1 = \rho u(t),$$

and

$$K(x, t) = \begin{cases} \rho \mathrm{v}(t) u(x), & [at] \\ \rho u(t) \mathrm{v}(x), & [tb]. \end{cases}$$

Now impose the condition D):

$$K'(t - 0) - K'(t + 0) = \frac{1}{p(t)},$$

that is

$$\rho \mathrm{v}(t)u'(t) - \rho u(t)\mathrm{v}'(t) = \frac{1}{p(t)}.$$

But

$$p(u\mathrm{v}' - \mathrm{v}u') = -1.$$

Therefore

$$\rho\left[\mathrm{v}(t)u'(t) - u(t)\mathrm{v}'(t)\right] = \frac{\rho}{p(t)},$$

whence

$$\rho = +1.$$

Therefore

$$(7) \qquad K(x, t) = \begin{cases} \mathrm{v}(t)u(x) \equiv K_0(x, t), & a \leq x \leq t \\ u(t)\mathrm{v}(x) \equiv K_1(x, t), & t \leq x \leq b. \end{cases}$$

This is *Green's function* and it is uniquely determined.

Corollary.—Green's function is symmetric. That is

$$K(x, t) = K(t, x).$$

To show this, compute $K(z_1, z_2)$ and $K(z_2, z_1)$ where

$$0 \leq z_1 < z_2 \leq b.$$

$$K(z_1, z_2) = \mathrm{v}(z_2)u(z_1),$$

for here we identify z_1 and x, z_2 and t; and, since $z_1 < z_2$, we use the first expression for K, namely, $K_0(x, t)$. Similarly

$$K(z_2, z_1) = u(z_1)\mathrm{v}(z_2).$$

A comparison shows that

$$K(z_1, z_2) = K(z_2, z_1),$$

that is,

$$K(x, t) = K(t, x).$$

74. Equivalence Between the Boundary Problem and a Homogeneous Linear Integral Equation. *a) Hilbert's Fundamental Theorem.*—Under the assumptions (H_1), (H_2) let us consider the non-homogeneous linear differential equation

$$L(u) + f = 0,$$

where f is supposed to be continuous on $\left[ab\right]$. We can then prove the following theorem:

Theorem III A.— *If F is of class C'' and $L(F) + f = 0$, $R_0(F) = 0$, $R_1(F) = 0$, then*

$$F(x) = \int_a^b K(x, t)f(t)dt.$$

Proof.—We have

$$L(F) = -f \text{ on } \Big[ab\Big], \text{ and}$$

$$L(K) = 0 \quad \text{on } \Big[at\Big], \Big[tb\Big] \text{ separately.}$$

Multiply the first by $-K$ and the last by F and add. We obtain

(8) $FL(K) - KL(F) = Kf$ on $\Big[at\Big], \Big[tb\Big]$ separately.

Apply Green's formula to the left-hand side of (8). We obtain

$$\frac{d}{dx}p(FK' - KF') = Kf. \tag{9}$$

Integrate (9) from a to t and from t to b. We have

$$\Big[p(FK' - KF')\Big]_a^{t-0} = \int_a^{t-0} K(x, t)f(x)dx$$

$$\Big[p(FK' - KF')\Big]_{t+0}^b = \int_{t+0}^b K(x, t)f(x)dx.$$

Add the last two equations. We obtain

$$\Big[p(FK' - KF')\Big]_{t+0}^{t-0} - \Big[p(FK' - KF')\Big]^{x=a} + \Big[p(FK' - KF')\Big]^{x=b} = \int_a^b K(x, t)f(x)dx, \tag{10}$$

since $K(x, t)$ is continuous at $x = t$. Now p, F, F', and K are continuous at $x = t$.

Hence $$- KF'\Big]_{t+0}^{t-0} = 0.$$

While

$$p(FK')\Big]_{t+0}^{t-0} = p(t-0)F(t-0)K'(t-0) - p(t+0)\,F(t+0)K'(t+0)$$

$$= p(t)F(t)\Big[K'(t-0) - K'(t+0)\Big]$$

which by $D)$

$$= p(t)F(t)\frac{1}{p(t)}.$$

Now write the second term in (10) as a determinant

$$p(a)\begin{vmatrix} F(a) & F'(a) \\ K(a) & K'(a) \end{vmatrix}.$$

If $A \neq 0$, then

$$\Big[p(FK' - F'K)\Big]^{x=a} = \frac{p(a)}{A}\begin{vmatrix} AF(a) + BF'(a) & F'(a) \\ AK(a) + BK'(a) & K'(a) \end{vmatrix}$$

$$= \frac{p(a)}{A}\begin{vmatrix} R_0(F) & F'(a) \\ R_0(K) & K'(a) \end{vmatrix} = 0,$$

since

$$R_0(F) = 0,\ R_0(K) = 0.$$

If $A = 0$, then $B \neq 0$ and, as before,

$$\Big[p(FK' - KF')\Big]^{x=a} = \begin{vmatrix} F(a) & R_0(F) \\ K(a) & R_0(K) \end{vmatrix}\frac{p(a)}{B} = 0.$$

Likewise

$$\Big[p(FK' - KF')\Big]^{x=b} = 0.$$

Therefore,

$$F(t) = \int_a^b K(x,\,t)f(x)dx.$$

Whence, interchanging x and t, on account of the symmetry of $K(x, t)$, we have

(11) $$F(x) = \int_a^b K(x,\,t)f(t)dt.$$

Corollary.—If $F(x)$ is of class C'' and $R_0(F) = 0, R_1(F) = 0$, and $f = -L(F)$, then

(12) $$F(x) = \int_a^b K(x,\,t)\Big[-L(F)\Big]dt.$$

Equation (11) is an integral equation of the *first* kind, with f as the function to be determined. Equation (12) shows us that (11) has a solution:

$$f = -L(F).$$

Theorem IIIB.—*If* $F(x) = \int_a^b K(x, t)f(t)dt$, *then F is of class* C'', $L(F) + f = 0$, $R_0(F) = 0$, $R_1(F) = 0$.

Proof. In order to form the derivatives of $F(x)$, we break the integral into two parts, inasmuch as K' is not continuous for $x = t$.

$$(13) \qquad F(x) = \int_a^x K_1(x, t)f(t)dt + \int_x^b K_0(x, t)f(t)dt$$

$$F'(x) = \int_a^x K_1'(x, t)f(t)dt + \int_x^b K_0'(x, t)f(t)dt + K_1(x, x)f(x) - K_0(x, x)f(x).$$

But $K_1(x, x) - K_0(x, x) = 0$, since K is continuous. Therefore

$$(14) \quad F'(x) = \int_a^x K_1'(x, t)f(t)dt + \int_x^b K_0'(x, t)f(t)dt.$$

$$F''(x) = \int_a^x K_1''(x, t)f(t)dt + \int_x^b K_0''(x, t)f(t)dt + K_1'(x, x)f(x) - K_0'(x, x)f(x).$$

But
$$K_1'(x, x) - K_0'(x, x) = \frac{-1}{p(x)}.$$

Therefore

$$(15) \quad F''(x) = \int_a^x K_1''(x, t)f(t)dt + \int_x^b K_0''(x, t)f(t)dt - \frac{f(x)}{p(x)}.$$

Multiply (13), (14), (15) by $q(x)$, $p'(x)$, $p(x)$ respectively and add. We get

$$L(F) = \int_a^x L(K_1)f(t)dt + \int_x^b L(K_0)f(t)dt - f(x).$$

But $L(K_1) = 0$ and $L(K_0) = 0$, therefore

(16) $$L(F) = -f(x).$$

Now form $R_0(F)$.

$$R_0(F) = AF(a) + BF'(a)$$
$$= \int_a^b R_0(K_0)f(t)dt = 0,$$

since $R_0(K_0) = 0$. Similarly, we show that $R_1(F) = 0$.

That $F(x)$ is of class C'' follows from (13), (14), and (15), if we substitute for K_0, K_1 their explicit expressions from (7).

We combine Theorems III*A* and III*B* into the following, which is Hilbert's third fundamental theorem:

Theorem III.—*If $f(x)$ is continuous, then*

$$F(x) = \int_a^b K(x, t)f(t)dt$$

implies and is implied by $F(x)$ is of class C'',

$$L(F) + f = 0,\ R_0(F) = 0, \text{ and } R_1(F) = 0.$$

b) Equivalence of Boundary Problem and Integral Equation.—If in Theorem III we put

$$f(x) = \lambda g(x)u(x),\ F(x) = u(x)$$

we obtain the theorem:

Theorem IV.—*If $u(x)$ is continuous, then*

(17) $$u(x) = \lambda \int_a^b K(x, t)g(t)u(t)dt$$

implies and is implied by $u(x)$ is of class C'',

$$L(u) + \lambda gu = 0,\ R_0(u) = 0, \text{ and } R_1(u) = 0.$$

For λ is constant and by (H_1) g is continuous, therefore the hypothesis that $f(x)$ is continuous becomes $u(x)$ is continuous.

We notice that (17) is a homogeneous linear integral equation of the second kind with, in general, an unsymmetric kernel:

(18) $$H(x, t) \equiv K(x, t)g(t).$$

We remark further that the condition that $u(x)$ be of class C'' carries with it the condition that $u(x)$ is of class C. Thus we drop the explicit statement that $u(x)$ be continuous and obtain from Theorem IV the two following theorems:

Theorem IVA.—*The conditions*

a) $u(x)$ is of class C''.

b) $L(u) + \lambda g u = 0,\ R_0(u) = 0,\ R_1(u) = 0$

imply that

$$u(x) = \lambda \int_a^b K(x, t) g(t) u(t) dt.$$

Theorem IVB.—*The conditions*

a) $u(x)$ is continuous.

b) $u(x) = \lambda \int_a^b K(x, t) g(t) u(t) dt$

imply that

a) $u(x)$ is of class C''.

b) $L(u) + \lambda g u = 0,\ R_0(u) = 0,\ R_1(u) = 0.$

Theorems *IVA* and *IVB* establish the equivalence of the boundary problem and of the integral equation. Hence,

1) If λ_0 is a characteristic constant of the boundary problem, then λ_0 is a characteristic constant of the integral equation and *vice versa*.

2) If $u(x)$ is a fundamental function for the boundary problem, belonging to λ_0, then $u(x)$ is a fundamental function, belonging to λ_0, for the integral equation and *vice versa.*

We remark that for this particular kernel (18), the fundamental functions are of class C''.

75. Special Case $\mathbf{g(x) \equiv 1}$.—For the special case $g(x) \equiv 1$, the equivalence between the boundary problem and the integral equation becomes:

$$(19)\qquad \begin{cases} L(u) + \lambda u = 0,\ R_0(u) = 0,\ R_1(u) = 0 \\ \text{implies and is implied by} \\ u(x) = \lambda \int_a^b K(x, t) u(t) dt. \end{cases}$$

The kernel of the integral equation in (19) is now symmetric and thus the results of both the Fredholm and the Hilbert-Schmidt theory apply.

Let us state some of the results of the Hilbert-Schmidt theory:

1) There exists at least one characteristic constant.
2) All of the characteristic constants are real.
3) The index q = the multiplicity r.
4) There exists a complete normalized orthogonal system of fundamental functions $\left\{ \psi_\nu(x) \right\}$ with corresponding characteristic constants $\left\{ \lambda_\nu \right\}$, $\nu = 1, 2, \ldots$.
5) If $\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu}$ is uniformly convergent on R, then

(20) $$\sum_\nu \frac{\psi_\nu(x)\psi_\nu(t)}{\lambda_\nu} = K(x, t).$$

6) If $u(x)$ is expressible in the form

$$u(x) = \lambda \int_a^b K(x, t)u(t)dt$$

and if $u(x)$ is continuous, then

(21) $$u(x) = \sum_\nu C_\nu \psi_\nu(x),$$

where $C_\nu = (u\psi_\nu)$, and the series is absolutely and uniformly convergent on $\left[ab \right]$.

For the special case under consideration, each of these six statements can be made for the boundary problem on account of the equivalence established.

Further results follow, for the kernel under consideration, besides satisfying the conditions $K(x, t)$, is continuous, $\not\equiv 0$, real, and symmetric of the Hilbert-Schmidt theory, satisfies now, in addition, the four conditions $A)$ $B)$ $C)$ $D)$ of §73.

Theorem V.—*There exists always an infinite number of characteristic constants* $\left\{ \lambda\nu \right\}$ *and a corresponding complete normalized orthogonal system of fundamental functions* $\left\{ \psi_\nu(x) \right\}$.

Proof.—Suppose that there is a finite number of characteristic constants. Then there are a finite number of fundamental functions:

$$\psi_1(x), \quad . \quad . \quad . \quad , \psi_m(x)$$
$$\lambda_1, \quad . \quad . \quad . \quad , \lambda_m.$$

For these the bilinear formula (20) holds. But $\psi_\nu(x)$ is of class C'' on R and, therefore, $K(x, t)$ would be of class C'' on R, which contradicts D), namely

$$K'(x, \ t - 0) - K'(x, t + 0) = \frac{1}{p(t)}.$$

Theorem VI.—*If* $u(x)$ *is of class* C'' *and* $R_0(u) = 0$,

$$R_1(u) = 0,$$

then

$$u(x) = \sum_\nu C_\nu \psi_\nu(x)$$

where $C_\nu = (u\psi_\nu)$, *and the series is absolutely and uniformly convergent.*

Proof.—From the corollary to Theorem IIIA

$$u(x) = \int_a^b K(x, t)\Big[- L(u) \Big] dx.$$

But $L(u)$ is continuous and, therefore, from (21)

$$u(x) = \sum_\nu C_\nu \psi_\nu(x).$$

Theorem VII.—*For every characteristic constant* λ *the index* q *is unity:* $q = 1$.

Proof.—Suppose $q > 2$, then at least three of the characteristic constants are equal, say, $\lambda_1 = \lambda_2 = \lambda_3$. Let the corresponding fundamental functions be $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$. We would then have three different independent solutions for $\lambda = \lambda_1$, of the same integral equation, that is, by the equivalence theorem three different solutions of the same differential equation of the second order. Hence one of them, say, ψ_3, is linearly expressible in terms of the other two:

$$\psi_3(x) = C_1\psi_1 + C_2\psi_2.$$

This is a contradiction. Therefore $q \ngtr 2$.

Suppose $q = 2$. Then there would be two fundamental functions $\psi_1(x)$, $\psi_2(x)$ belonging to $\lambda = \lambda_1$. Now, by Green's formula,

$$p\Big[\psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x)\Big] = C \neq 0. \tag{22}$$

But

$$R_0(\psi_1) = A\psi_1(a) + B\psi_1'(a) = 0$$
$$R_0(\psi_2) = A\psi_2(a) + B\psi_2'(a) = 0.$$

These are two linear homogeneous equations for the determination of A, B, not both zero. Hence

$$\psi_1(a)\psi_2'(a) - \psi_2(a)\psi_1'(a) = 0,$$

which contradicts (22), therefore $q \neq 2$. Therefore we have $q = 1$.

We shall now discuss the more special boundary problem in which it follows from the boundary conditions that

$$\Big[puu'\Big]_a^b = 0. \tag{23}$$

For instance, if we have

$$u(a) = 0,\ u(b) = 0, \text{ or } u(a) = 0,\ u'(b) = 0,$$
or $$u'(a) = 0,\ u(b) = 0, \text{ or } u'(a) = 0,\ u'(b) = 0,$$
equation (23) is satisfied.

For this more special boundary problem we have the theorem:

Theorem VIII.—*If, as a consequence of the boundary conditions, we have*

$$\Big[puu'\Big]_a^b = 0,$$

then there exists a $\begin{Bmatrix} \textit{smallest} \\ \textit{largest} \end{Bmatrix}$ *characteristic constant if* $\begin{Bmatrix} p(x) > 0 \\ p(x) < 0 \end{Bmatrix}$.

Proof.—The assumptions $p(x) > 0$, $p(x) < 0$ are in harmony with our previous assumption that $p(x) \neq 0$ on $\Big[ab\Big]$, for then $p(x)$ is always of the same sign. We have previously shown that there are an infinite number of characteristic constants, but, so far as we know, at present they may be infinite in number in both the positive and negative directions. Let $\psi_\nu(x)$ be a fundamental function belonging to λ_ν, then

$$\text{(24)} \qquad \begin{aligned} L(\psi_\nu) + \lambda_\nu\psi_\nu &= 0, \\ R_0(\psi_\nu) = 0,\ R_1(\psi_\nu) &= 0, \end{aligned}$$

ψ_ν is of class C'', and $\psi_\nu \not\equiv 0$. Multiply both members of (24) by ψ_ν and integrate. We obtain

$$\text{(25)} \qquad \lambda_\nu = -\int_a^b \psi_\nu(x)L(\psi_\nu)dx$$

since

$$\int_a^b \psi_\nu^{\,2}(x)dx = 1.$$

In (25) put the explicit expression for $L(\psi_\nu)$. We have

$$\lambda_\nu = -\int_a^b \Big[\psi_\nu \frac{d}{dx}(p\psi_\nu') + q\psi_\nu^{\,2}\Big]dx,$$

or

$$\lambda_\nu = -\int_a^b \psi_\nu \frac{d}{dx}(p\psi_\nu')dx + \int_a^b (-q)\psi_\nu^{\,2}dx.$$

Integrate the first integral by parts. We find

$$\lambda_\nu = -\Big[p\psi_\nu\psi_\nu'\Big]_a^b + \int_a^b p\psi_\nu'^2dx + \int_a^b (-q)\psi_\nu^2dx.$$

Now $-q$ is a continuous function on $\Big[ab\Big]$ and hence has a finite maximum M and a finite minimum m:

$$m \leqq -q \leqq M \text{ on } \Big[ab\Big].$$

Therefore

$$(26) \qquad \begin{cases} \lambda_\nu \leqq \int_a^b p\psi_\nu'^2dx + M, \text{ and} \\ \lambda_\nu \geqq \int_a^b p\psi_\nu'^2dx + m. \end{cases}$$

If $p > 0$ on $\Big[ab\Big]$, then the integral in (26) is a positive number and therefore $\lambda_\nu > m$.

If $p < 0$ on $\Big[ab\Big]$, then the integrand in (26) is a negative number and therefore $\lambda_\nu < M$.

Combining these results with the fact that a finite interval can contain only a finite number of the λ_ν's, we obtain the result:

When $p(x) > 0$, the characteristic constants can be arranged in a sequence increasing towards plus infinity:

$$p(x) > 0; \lambda_1 < \lambda_2 < \lambda_3 < \quad . \quad . \quad . \quad < \lambda_\nu < \quad . \quad . \quad .$$

and when $p(x) < 0$, they may be arranged in a sequence decreasing towards minus infinity:

$$p(x) < 0; \lambda_1 > \lambda_2 > \lambda_3 > \quad . \quad . \quad . \quad > \lambda_\nu > \quad . \quad . \quad .$$

Corollary.—If $p(x) > 0$ and $q \leqq 0$ on $\Big[ab\Big]$, then all of the characteristic constants are positive; if $p(x) < 0$ and $q \geqq 0$ on $\Big[ab\Big]$, then all of the characteristic constants are negative.

For if $-q \geqq 0$, then $m \geqq 0$, so that, from (26), since $p > 0$, we get

$$m + (a \text{ positive quantity}) \leqq \lambda_\nu.$$

Therefore $$\lambda_\nu > 0.$$

If $-q \leqq 0$, then $M \leqq 0$, so that, from (26), since $p < 0$, we obtain

$$\lambda_\nu \leqq M + (\text{a negative quantity not zero}).$$

Therefore $$\lambda_\nu < 0.$$

Definition.—A real symmetric kernel $K(x, t)$ is said to be **closed** *if there exists no continuous function $h(x)$ other than $h(x) \equiv 0$, for which*

$$\int_a^b K(x, t)h(t)dt = 0, (x).$$

Theorem IX.—*Green's function $K(x, t)$ for our boundary problem is always closed.*

Proof.—We make use of Hilbert's third fundamental theorem, which states the equivalence of the boundary problem and the homogeneous linear integral equation:

If $f(x)$ is continuous, then

$$F(x) = \int_a^b K(x, t)f(t)dt$$

implies and is implied by $F(x)$ is of class C'',

$$L(F) + f = 0, R_0(F) = 0, \text{ and } R_1(F) = 0.$$

We apply this theorem for $F(x) \equiv 0$, whence if $f(x)$ is continuous and

$$\int_a^b K(x, t)f(t)dt = 0, (x),$$

then $$f = -L(0) \equiv 0.$$

76. Miscellaneous Remarks. *a) The General Case $g(x) \not\equiv 1$.*—In the previous article we considered the special case $g(x) \equiv 1$. We now consider the problem for $g(x) \not\equiv 1$. Hilbert's third fundamental theorem now reads:

If $u(x)$ is continuous, then

$$u(x) = \lambda \int_a^b K(x, t)g(t)u(t)dt$$

implies and is implied by $u(x)$ is of class C'',

$$L(u) + \lambda g u = 0,\ R_0(u) = 0,\ R_1(u) = 0.$$

There are two cases to be considered:

Case I.—$g(x) \neq 0$ *on* $\left[ab\right]$. This case can be reduced to the case $g(x) \equiv 1$. Since $g(x) \neq 0$ on $\left[ab\right]$, we have either $g(x) > 0$ or $g(x) < 0$ on $\left[ab\right]$. Consider the first case. Multiply both sides of

$$u(x) = \lambda \int_a^b K(x, t)g(t)u(t)dt \tag{27}$$

by $\sqrt{g(x)}$, and put

$$\bar{u}(x) = \sqrt{g(x)} \cdot u(x).$$
$$\bar{K}(x, t) = K(x, t)\sqrt{g(x)}\sqrt{g(t)}.$$

We then obtain

$$\bar{u}(x) = \lambda \int_a^b \bar{K}(x, t)\bar{u}(t)dt. \tag{28}$$

This is the reduction desired, for $\bar{K}(x, t)$ is symmetric.

If $g(x) < 0$ on $\left[ab\right]$, multiply both sides of (27) by $\sqrt{-g(x)}$, and put

$$\bar{u}(x) = \sqrt{-g(x)} \cdot u(x).$$
$$\bar{K}(x, t) = K(x, t)\sqrt{-g(x)}\sqrt{-g(t)}.$$

We obtain

$$\bar{u}(x) = -\lambda \int_a^b \bar{K}(x, t)\bar{u}(t)dt,$$

which is again the reduction desired.

Case II.—$g(x)$ vanishes at some point of $\left[ab\right]$. This case has been treated by Hilbert ["Gott. Nach.," 5, p. 462, 1906] for functions $K(x, t)$ which are *definite*, by means of the theory of quadratic forms with infinitely many variables. J. Marty [*Comptes Rendus*, vol. 150, p. 515; *Ibid.*, p. 605, 1910] has reached the same results without the use of quadratic forms of infinitely many variables.

Definition.—A real symmetric[1] *kernel $K(x, t)$ is said to be definite if no continuous function exists, other than $h(x) \equiv 0$, for which*

$$\int_a^b \int_a^b K(x, t)h(x)h(t)dxdt = 0.$$

The name "definite" has been used on account of the analogy with a *definite* quadratic form $\sum K_{ij}y_iy_j$ which vanishes only when all of the y's vanish.

Integral equations of the form

$$(29) \qquad u(x) = f(x) + \lambda \int_a^b K(x, t)g(t)u(t)dt$$

with $K(x, t)$ *definite* have been called by Hilbert *polar* integral equations or integral equations of the third kind. If we multiply the equation (29) by $g(x)$ and put

$$\bar{f}(x) = f(x)g(x)$$
$$\bar{K}(x, t) = K(x, t)g(x)g(t),$$

we obtain the equation in the form in which Hilbert considered it:

$$g(x)u(x) = \bar{f}(x) + \lambda \int_a^b \bar{K}(x, t)u(t)dt.$$

[1] For a discussion of this and other special kernels consult LALESCO, T., "Théorie Des Équations Intégrales," pp. 64*ff*.

Definition.—A kernel $H(x, t)$ is said to be symmetrizable if there exists a definite symmetric kernel $G(x, t)$ such that either

$$K_1(x, t) = \int_a^b G(x, s)H(s, t)ds$$

or

$$K_2(x, t) = \int_a^b H(x, s)G(s, t)ds$$

is symmetrical.

In this instance the kernel

$$H(x, t) = K(x, t)g(t)$$

is symmetrizable if $K(x, t)$ is definite, for

$$K_2(x, t) = \int_a^b K(x, s)g(s)K(s, t)ds$$

is symmetric, since

$$\begin{aligned} K_2(t, x) &= \int_a^b K(t, s)g(s)K(s, x)ds \\ &= \int_a^b K(s, t)g(s)K(x, s)ds \\ &= K_2(x, t). \end{aligned}$$

For symmetrizable kerns Marty has shown that

1) There exists at least one characteristic constant.
2) All of the characteristic constants are real.

b) Non-homogeneous Boundary Problems.—We next consider the non-homogeneous boundary problem:

$$L(u) + \lambda g u + r = 0, \qquad R_0(u) = 0, \qquad R_1(u) = 0,$$

where r is a given function of x, continuous on $\left[ab\right]$, and $R_0(u)$, $R_1(u)$ are defined by (4).

We are going to show that the boundary problem is equivalent to a non-homogeneous integral equation. To do this we again make use of Hilbert's third fundamental theorem for

$$f = \lambda gu + r, F = u.$$

Since λ, g, r are continuous by hypothesis, this theorem now reads: if $u(x)$ is continuous, then the combined statement u is of class C'', $L(u) + \lambda gu + r = 0$, $R_0(u) = 0$, $R_1(u) = 0$ is equivalent to the statement that

$$u(x) = \int_a^b K(x, t)\Big[\lambda g(t)u(t) + r(t)\Big]dt$$
$$= \lambda\int_a^b K(x, t)g(t)u(t)dt + \int_a^b K(x, t)r(t)dt.$$

Hence we have the theorem:

Theorem X.—*The non-homogeneous boundary problem:*

$$L(u) + \lambda gu + r = 0, R_0(u) = 0, R_1(u) = 0$$

with the further condition that $u(x)$ is of class C'', is equivalent to the non-homogeneous integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x, t)g(t)u(t)dt$$

where $u(x)$ is continuous and $f(x) = \int_a^b K(x, t)r(t)dt.$

c) *The Exceptional Case $\lambda = 0$ Is a Characteristic Constant.*—All of the preceding developments have been made under the hypothesis (H_2) that $\lambda = 0$ is not a characteristic constant of the boundary problem. This hypothesis, however, is not satisfied in certain problems of mathematical physics. For example, the following boundary problem in the theory of heat:

$$\frac{d^2u}{dx^2} + \lambda u = 0, u'(a) = 0, u'(b) = 0$$

has the non-trivial solution u = constant for $\lambda = 0$.

The exceptional case in which the hypothesis (H_2) is not satisfied can be treated, according to Hilbert,[1] by introducing a modified Green's function.

[1] See HILBERT, "Gott. Nach.," p. 213, 1904.

In most cases, however, the following simple artifice, due to Kneser will be sufficient.

Let us replace the assumption (H_2) by the much milder assumption (H_2'):

(H_2') *There exists at least one value c of λ which is not a characteristic constant for the boundary problem.*

Let us write the differential equation

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + (q + \lambda g)u = 0$$

in the form

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + \left[q + cg + (\lambda - c)g\right]u = 0$$

or

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + (\bar{q} + \bar{\lambda}g)u = 0,$$

where

$$\bar{q} = q + cg,\ \bar{\lambda} = \lambda - c.$$

Then $\bar{\lambda} = 0$ is certainly not a characteristic constant of the boundary problem

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + (\bar{q} + \bar{\lambda}g)u = 0,\ R_0(u) = 0,\ R_1(u) = 0,$$

since, by hypothesis (H_2'), the boundary problem

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu + cgu = 0,\ R_0(u) = 0,\ R_1(u) = 0$$

has no solution other than $u \equiv 0$.

II. Applications to Some Problems of the Calculus of Variations

77. Some Auxiliary Theorems of the Calculus of Variations. *a) Formulation of the Simplest Type of Problem.*—For the simplest type of problems of the calculus of variations we have given

1) Two points $P_0(x_0, y_0)$, $P_1(x_1, y_1)$.
2) A function $F(x, y, y')$ of three independent variables.

Required: to find among all curves

(30) $$y = y(x)$$

joining P_0 and P_1 that one which furnishes for the definite integral

$$J(y) = \int_{x_0}^{x_1} F\left[x, y(x), y'(x)\right]dx, \left[y'(x) = \frac{d}{dx}y(x)\right]$$

the smallest value.

Concerning the admissible curves (30) we make the assumption that they satisfy the following conditions:

(31)
A) $y(x)$ of class C''.
B) $y(x_0) = y_0$; $y(x_1) = y_1$.

We assume that the function F is of class C''' for all systems of values x, $y(x)$, $y'(x)$ furnished by all of the admissible curves.

b) Euler's Differential Equation.—Suppose we have found the minimizing curve C_0

C_0: $$y = f(x).$$

We replace it by a neighboring admissible curve of the special form

C: $$y = f(x) + \epsilon\eta(x),$$

where ϵ is a small constant and $\eta(x)$ a function of x satisfying the conditions:

(32)
A′) $\eta(x)$ is of class C''.
B′) $\eta(x_0) = 0$, $\eta(x_1) = 0$.

Since C_0 minimizes the integral, we have

$$\int_{x_0}^{x_1} F\left[x, f + \epsilon\eta, f' + \epsilon\eta'\right]dx \geq \int_{x_0}^{x_1} F\left[x, f, f'\right]dx,$$

which we shall write in the form

$$I(\epsilon) \geq I(0).$$

Considered as a function of ϵ, $I(\epsilon)$ has, therefore, a minimum for $\epsilon = 0$ and hence

$$I'(0) = 0, \quad I''(0) \geqq 0.$$

These are *necessary* conditions for a minimum. It is customary to write

$$\delta I \equiv \epsilon I'(0), \qquad \delta^2 I \equiv \epsilon^2 I''(0), \ (\eta)$$

and to call $\delta I, \delta^2 I$, the first and second variation respectively.

We first consider the condition $I'(0) = 0$. By definition

$$I(\epsilon) = \int_{x_0}^{x_1} F\left[x, y + \epsilon\eta, y' + \epsilon\eta'\right] dx,$$

whence, by the rules for differentiating a definite integral with respect to a parameter,

$$I'(\epsilon) = \int_{x_0}^{x_1} \left[\bar{F}_y \eta + \bar{F}_{y'} \eta'\right] dx, \ (\eta) \tag{33}$$

where the dash indicates that we use the arguments, $x, f(x) + \epsilon\eta(x), f'(x) + \epsilon\eta'(x)$. Therefore

$$I'(0) = \int_{x_0}^{x_1} \left(F_y \eta + F_{y'} \eta'\right) dx, \ (\eta)$$

the arguments now being $x, f(x), f'(x)$.[1]

An integration by parts gives

$$\begin{aligned} I'(0) &= \left[\eta F_{y'}\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta \left(F_y - \frac{d}{dx} F_{y'}\right) dx, \ (\eta) \\ &= \int_{x_0}^{x_1} \eta \left(F_y - \frac{d}{dx} F_{y'}\right) dx, \ (\eta) \end{aligned}$$

on account of (32).

But $I'(0) = 0$, and, hence, by the fundamental theorem of the calculus of variations,[2]

$$F_y - \frac{d}{dx} F_{y'} = 0. \tag{34}$$

[1] Consult Bolza, "Lectures on the Calculus of Variations," pp. 16*ff*., University of Chicago Press, 1904.

[2] See Bolza, "Lectures on the Calculus of Variations," §5.

This is a differential equation of the second order for the determination of $y = f(x)$. It is the first necessary condition and is known as Euler's equation. Its solution involves two arbitrary constants which have to be determined by the two conditions (31).

From (33) we get

$$f''(\epsilon) = \int_{x_0}^{x_1}\left(\bar{F}_{yy}\eta^2 + 2\bar{F}_{yy'}\eta\eta' + \bar{F}_{y'y'}\eta'^2\right)dx,$$

whence

$$f''(0) = \int_{x_0}^{x_1}\left(F_{yy}\eta^2 + 2F_{yy'}\eta\eta' + F_{y'y'}\eta'^2\right)dx, \ (\eta).$$

c) *Euler's Rule for Isoperimetric Problems.*—For isoperimetric problems the admissible curves are subject to a third condition C in addition to A) and B):

$$C: \qquad \int_{x_0}^{x_1} G\left[x, y(x), y'(x)\right] = l, \text{ a constant.}$$

The problem is, then, to determine, among all curves

$$y = y(x)$$

satisfying the conditions A) B) C), that one which will furnish the smallest value for the integral

$$J(y) = \int_{x_0}^{x_1} F(x, y, y')dx.$$

The conditions on G are the same as those on F, *viz.*, F and G are continuous and possess continuous partial derivatives of the first, second, and third order in the region under consideration. Put

$$H = F + \lambda G,$$

λ an arbitrary constant. Then, by Euler's rule, the first necessary condition for a minimum is the same as if it were required to minimize the integral

$$\int_{x_0}^{x_1} H(x, y, y')dx$$

with respect to the totality of curves satisfying the conditions A) and B), that is,

$$H_y - \frac{d}{dx}H_{y'} = 0,$$

or, more explicitly,

$$F_y - \lambda G_y - \frac{d}{dx}(F_{y'} - \lambda G_{y'}) = 0.$$

The solution of this equation involves three arbitrary constants (α, β, λ) which are to be determined by the three conditions B) and C).

78. Dirichlet's Problem.—We now propose to minimize the integral

$$(35) \qquad D(u) = \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - qu^2 \right] dx$$

with respect to the totality M of curves satisfying the conditions

$$(36) \qquad \begin{cases} u \text{ is of class } C'', \ u(a) = 0, \ u(b) = 0, \\ \displaystyle\int_a^b u^2(x)dx = 1, \end{cases}$$

with the further hypotheses

$$\left. \begin{array}{c} p > 0, \ p \text{ of class } C' \\ q \text{ continuous} \end{array} \right\} \text{ on } \left[ab \right].$$

Hilbert, to whom the following developments are due, calls this problem *Dirichlet's problem*. The problem is an isoperimetrical problem of the type considered in §77, with

$$F = pu'^2 - qu^2, \ G = u^2.$$

Hence

$$H(x, u, u') = pu'^2 - (q + \lambda)u^2$$

and, therefore, Euler's differential equation for H is

$$H_u - \frac{d}{dx}H_{u'} = -2(q + \lambda)u - \frac{d}{dx}(2pu') = 0.$$

Whence

$$\frac{d}{dx}(pu') + (q + \lambda)u = 0,$$

which may be written

(37) $$L(u) + \lambda u = 0.$$

Every solution of this problem must satisfy this equation and the conditions (36). Now (36) and (37) constitute a boundary problem of the type previously discussed in which

$$g(x) \equiv 1, \left[puu' \right]_a^b = 0,$$

hence the Theorems I to IX hold.

We can, therefore, state at once that this problem (36), (37) has no solutions except when λ is a characteristic constant. We know that

1) The characteristic constants are real.
2) The characteristic constants are infinite in number.
3) For each characteristic constant the index $q = 1$.

Therefore the λ's are distinct, and, since $p > 0$, we have, by Theorem VIII,

(38) $$\lambda_1 < \lambda_2 < \lambda_3 < \cdot \cdot \cdot ,$$

with a corresponding complete set,

$$\psi_1(x), \psi_2(x), \psi_3(x), \cdot \cdot \cdot$$

of normalized orthogonal fundamental functions of class C''. We have, then, for $n = 1, 2, \cdot \cdot \cdot$

$$L(\psi_n) + \lambda_n \psi_n = 0,$$

$$\psi_n(a) = 0, \psi_n(b) = 0,$$

(39) $$\int_a^b \psi_n^2(x)dx = 1.$$

If, then, Dirichlet's problem has a solution, it must be of the form

$$u = C\psi_n(x), \lambda = \lambda_n.$$

But from (36) we obtain

$$C^2 \int_a^b \psi_n^2(x)dx = 1.$$

Comparing with (39) we find that $C = \pm 1$. The only possible solutions, therefore, are

$$u = \pm \psi_n(x), \lambda = \lambda_n.$$

But in the proof of Theorem VIII we showed that

$$\lambda_n = \int_a^b (p\psi_n'^2 - q\psi_n^2)dx.$$

Therefore

$$\lambda_n = D(\pm \psi_n), \text{ by (35).}$$

Hence, it follows from (38) that

$$D(\pm \psi_1) < D(\pm \psi_n), \ (n = 2, 3, \ . \ . \ . \ ).$$

Hence we infer that: *If there exists at all a function which minimizes $D(u)$ with respect to the totality M of all admissible curves, it must be the function*

$$u = \pm \psi_1(x)$$

and λ must have the value λ_1.

b) Sufficiency Proof. *α) Transformation of $D(u)$.*—By an integration by parts we obtain

$$\int_a^b (pu')u'dx = upu' \Big]_a^b - \int_a^b u\frac{d}{dx}\left(p\frac{du}{dx}\right)dx.$$

But $$upu' \Big]_a^b = 0, \text{ since } u(a) = 0, u(b) = 0.$$

Therefore

$$D(u) \equiv \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - qu^2\right]dx$$
$$= -\int_a^b u\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right]dx.$$

Therefore

$$D(u) = -\int_a^b uL(u)dx.$$

Let us put

$$-L(u) = \omega(x).$$

Then ω is continuous and we have

$$D(u) = +\int_a^b u(x)\omega(x)dx.$$

We now apply Hilbert's third fundamental theorem, whence from

$$u \text{ is of class } C'', \; L(u) + \omega = 0, \; u(a) = 0, \; u(b) = 0$$

it follows that

$$u(x) = \int_a^b K(x, t)\omega(t)dt.$$

Multiply both members of this equation by $\omega(x)$ and integrate. We obtain

$$(40) \qquad D(u) = \int_a^b \int_a^b K(x, t)\omega(t)\omega(x)dtdx.$$

β) *Applications of the Expansion Theorem.*—From Corollary II to Theorem XVII of §68, we have

$$u(x) = \int_a^b K(x, t)\omega(t)dt = \sum_{\nu=1}^{\infty} \frac{(\omega\psi_\nu)}{\lambda_\nu} \psi_\nu(x),$$

the series being absolutely and uniformly convergent. Let us put

$$C_\nu = (\omega\psi_\nu), \text{ constant},$$

then

$$(41) \qquad u(x) = \sum_{\nu=1}^{\infty} \frac{C_\nu}{\lambda_\nu} \psi_\nu(x).$$

Form

$$u^2(x) = \sum_{\mu,\nu} \frac{C_\mu C_\nu}{\lambda_\mu \lambda_\nu} \psi_\mu(x)\psi_\nu(x).$$

This series is also absolutely and uniformly convergent. We obtain, therefore,

$$\int_a^b u^2(x)dx = \sum_{\mu,\nu} \frac{C_\mu C_\nu}{\lambda_\mu \lambda_\nu} \int_a^b \psi_\mu(x)\psi_\nu(x)dx = 1.$$

But
$$\int_a^b \psi_\mu(x)\psi_\nu(x)dx = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu. \end{cases}$$

Therefore

(42)
$$\sum_\nu \frac{C_\nu^2}{\lambda_\nu^2} = 1.$$

We now apply Corollary III to Theorem XVII of §68, to the double integral in (40). We obtain

$$D(u) = \sum_{\nu=1}^{\infty} \frac{(\omega\psi_\nu)^2}{\lambda_\nu} = \sum_{\nu=1}^{\infty} \frac{C_\nu^2}{\lambda_\nu}.$$

γ) *Computation of* $D(u) - D(\pm\,\psi_1)$.—Let us now compute the difference

$$D(u) - D(\pm\,\psi_1) = D(u) - \lambda_1.$$

On account of (42)

$$\lambda_1 = \sum_{\nu=1}^{\infty} \frac{C_\nu^2 \lambda_1}{\lambda_\nu^2}.$$

Therefore

$$D(u) - \lambda_1 = \sum_\nu \frac{C_\nu^2}{\lambda_\nu} - \sum \frac{C_\nu^2 \lambda_1}{\lambda_\nu^2}$$

$$= \sum_{\nu=2}^{\infty} \frac{C_\nu^2}{\lambda_\nu^2}(\lambda_\nu - \lambda_1) \geqq 0,$$

since
$$\frac{C_\nu^2}{\lambda_\nu^2} \geqq 0, \text{ and } \lambda_\nu - \lambda_1 > 0 \text{ for } \nu \geqq 2.$$

Therefore we have

$$D(u) - D(\pm\,\psi_1) \geqq 0.$$

The equality holds only if

$$C_\nu = 0 \text{ for } \nu = 2, 3, \ . \ . \ . \ , \infty.$$

But in this event from (42) we obtain $C_1 = \pm \lambda_1$, whence from (41), $u(x) = \pm \psi_1(x)$. Therefore, whenever $u(x) \not\equiv \psi_1(x)$, it results that

$$D(u) - D(\pm \psi_1(x)) \geqq 0.$$

Thus we have proved the following theorem:

Theorem XI.—$u = \pm \psi_1(x)$ *furnishes for*

$$D(u) = \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - qu^2 \right] dx$$

a smaller value (viz., λ_1*) than any other function satisfying the conditions*

$$u \text{ is of class } C'', u(a) = 0, u(b) = 0, \int_a^b u^2(x)dx = 1.$$

c) Dropping the Assumption (H_2).—The results for the Dirichlet problem seem to presuppose (H_2), that is, that $\lambda = 0$ is not a characteristic constant of the boundary problem

$$L(u) + \lambda u = 0, u(a) = 0, u(b) = 0,$$

but they are independent of this assumption.

Proof.—α) We notice first that the assumption, $\lambda = 0$ is not a characteristic constant, was necessary for the construction of a Green's function. But the Green's function did not occur in the proof of Theorem VIII. In this theorem, we made the special assumption that

$$R_0(u) = 0, R_1(u) = 0, \text{ imply } \Big[puu' \Big]_a^b = 0.$$

Let then λ_0 be any characteristic constant of the boundary problem:

$$(43) \qquad L(u) + \lambda u = 0, R_0(u) = 0, R_1(u) = 0$$

and $\psi_0(x)$ a corresponding normalized fundamental function of the boundary problem:

ψ_0 of class C'', $L(\psi_0) + \lambda_0\psi_0 = 0$, $R_0(\psi_0) = 0$, $R_1(\psi_0) = 0$,

$$\int_a^b \psi_0^2(x)dx = 1.$$

Hence we derived, since $\left[p\psi_0\psi_0'\right]_a^b = 0$, without using the Green's function, that is, independently of (H_2) the equality

$$\lambda_0 = \int_a^b (p\psi_0'^2 - q\psi_0^2)dx \equiv D(\psi_0.)$$

Hence followed from the corollary to Theorem VIII that

$$p > 0,\ q \leqq 0, \text{ imply } \lambda_0 > 0;$$

that is, all of the characteristic constants of the boundary problem are greater than zero. Thus the hypothesis (H_2) is satisfied and the Theorems V to VIII certainly hold.

β) Suppose now that the condition $q \leqq 0$ is not satisfied. Denote, as before, by m the minimum of $-q$ on $\left[ab\right]$, that is,

$$q + m \leqq 0.$$

Then the differential equation

$$L(u) + \lambda u \equiv \frac{d}{dx}\left(p\frac{du}{dx}\right) + (q + \lambda)u = 0$$

becomes

$$\bar{L}(u) + \bar{\lambda}u \equiv \frac{d}{dx}\left(p\frac{du}{dx}\right) + \bar{q}u + \lambda u = 0$$

where

$$q + m = \bar{q} \text{ and } \lambda - m = \bar{\lambda}.$$

Since $p > 0$, $\bar{q} \leqq 0$, the Theorems V to VIII hold for the boundary problem

(44) $\bar{L}(u) + \bar{\lambda}u = 0,\ R_0(u) = 0,\ R_1(u) = 0.$

In particular, it has a single infinitude of real, distinct characteristic constants:

$$0 < \bar{\lambda}_1 < \bar{\lambda}_2 < \bar{\lambda}_3 < \ . \ . \ .$$

with corresponding normalized fundamental functions

$$\psi_1(x), \psi_2(x), \psi_3(x), \quad . \quad . \quad .$$

But since

$$\bar{L}(u) = L(u) + mu$$

it follows that, if λ_0 is a characteristic constant and $\psi_0(x)$ a fundamental function for (43), then $\bar{\lambda}_0 = \lambda_0 - m$ is a characteristic constant and $\psi_o(x)$ the corresponding fundamental function for (44), and *vice versa.*

Hence also the boundary problem (43) has an infinitude of real, distinct characteristic constants forming an increasing sequence:

$$\lambda_1 = \bar{\lambda}_1 + m, \lambda_2 = \bar{\lambda}_2 + m, \quad . \quad . \quad . \quad ,$$
$$\lambda_1 < \lambda_2 < \lambda_3 < \quad . \quad . \quad . \quad ,$$

with the corresponding normalized orthogonal fundamental functions $\psi_1(x)$, $\psi_2(x)$, ...

γ) Consider now the problem

$$D(u) = \text{minimum on } (M).$$

This problem is evidently equivalent to

$$\bar{D}(u) \equiv D(u) - m = \text{minimum on } (M), \text{ i.e.,}$$

$$\bar{D}(u) = \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - (q + m)u^2 \right] dx = \text{minimum on } (M).$$

Since $q + m \leqq 0$, our former results hold and

$$u = \pm \psi_1(x)$$

furnishes the minimum for $\bar{D}(u)$, that is,

$$\bar{D}\left[\pm \psi_1(x) \right] = \bar{\lambda}_1.$$

On account of the equivalence of the two problems, the same function $u = \pm \psi_1(x)$ furnishes the minimum for $D(u)$ and

$$D(\pm \psi_1) = \bar{D}(\pm \psi_1) + m = \bar{\lambda}_1 + m = \lambda_1.$$

But by β) $\lambda_1 = \bar{\lambda}_1 + m$ is the smallest characteristic constant for the boundary problem

$$L(u) + \lambda u = 0,\ u(a) = 0,\ u(b) = 0.$$

This shows that Theorem XI is true also when the assumption (H_2) is not satisfied.

79. Applications to the Second Variation.—Hilbert has made an application of the preceding result to the discussion of the second variation for the simplest type of variation problem considered in §77.

a) Reduction of the Problem.—The problem is to find the condition under which

$$(45)\qquad \int_{x_0}^{x_1}\left(F_{yy}\eta^2 + 2F_{yy'}\eta\eta' + F_{y'y'}\eta'^2\right)dx \geqq 0$$

for all functions $\eta(x)$ satisfying the conditions

A') $\eta(x)$ is of class C''.
B') $\eta(x_0) = 0,\ \eta(x_1) = 0.$

The arguments of $F_{yy}, F_{yy'}, F_{y'y'}$ are $x,\ y = f(x),\ y' = f'(x)$ where $f(x)$ is a solution of Euler's differential equation (34) satisfying the initial conditions. We adopt the notation

$$F_{yy}\left[x, f(x), f'(x)\right] = P(x),\ F_{yy'}\left[x, f(x), f'(x)\right] = Q\ (x),$$

$$F_{y'y'}\left[x, f(x), f'(x)\right] = R(x).$$

Then (45) becomes

$$(46)\qquad \int_{x_0}^{x_1}\left[P\eta^2 + 2Q\eta\eta' + R\eta'^2\right]dx \geqq 0,\ (\eta).$$

It is easily shown[1] that a first necessary condition for the inequality (46) is that $R \geqq 0$ on $\left[ab\right]$, which is Legendre's condition.

[1] Compare Bolza, "Lectures on the Calculus of Variations," §11.

We suppose this condition to be satisfied in the stronger form $R > 0$ on $\left[ab\right]$.

We now transform the integral by integrating the second term by parts

$$\int_{x_0}^{x_1} 2Q\eta\eta' dx = \int_{x_0}^{x_1} Q \frac{d}{dx} \eta^2 dx$$
$$= \eta^2 \Big]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta^2 Q' dx$$
$$= -\int_{x_0}^{x_1} \eta^2 Q' dx.$$

Therefore (45) becomes

$$\int_{x}^{x_1} \left[(P - Q')\eta^2 + R\eta'^2 \right] dx \geqq 0, \ (\eta).$$

If for $\qquad \eta, \ x_0, \ x_1, \ R, \ P - Q'$

we put $\qquad u, \ a, \ b, \ p, \ \ -q,$

then the above inequality becomes

$$D(u) \geqq 0$$

for the totality N of all functions u satisfying the conditions

$$u \text{ is of class } C'', \ u(a) = 0, \ u(b) = 0.$$

b) *Connection with Dirichlet's Problem.*—We now show the equivalence of the two statements

(47) $\qquad D(u) \geqq 0$, for all curves of class N, and

(48) $\qquad D(u) \geqq 0$, for all curves of class M.

A) The class of curves M is contained within the class N and hence

$$D(u) \geqq 0, \ (N) \text{ implies } D(u) \geqq 0, \ (M).$$

B) Suppose $u \not\equiv 0$ belongs to the class (N). Construct $u_1 = \rho u$ such that

$$\int_a^b u_1^2 \, dx = 1.$$

Then u_1 belongs to M, for u_1 is of class C'' and $u_1(a) = 0$, $u_1(b) = 0$, since $u(a) = 0$, $u(b) = 0$ and by construction

$$\int_a^b u_1^2 dx = 1.$$

Therefore $$D(u_1) \geqq 0.$$

But $$D(u_1) = D(\rho u) = \rho^2 D(u).$$

Therefore $$D(u) \geqq 0.$$

Hence, if u belongs to the class (N) and $u \not\equiv 0$, then $D(u) \geqq 0$, while if $u \equiv 0$ also $D(u) = 0$. Therefore, if u belongs to (N), then $D(u) \geqq 0$.

The equivalence between the two inequalities (47) and (48) being established, we can now apply the results of §78:

The smallest value which $D(u)$ can take in M is λ_1 and this value is furnished by $u = \pm \psi_1(x)$ and by no other function of M.

Hence if

1) $\lambda_1 > 0$, then $D(u) > 0$ (M).
2) $\lambda_1 = 0$, then $D(u) > 0$ (M).
   except when $u = \pm \psi_1(x)$, in which case $D(u) = 0$.
3) $\lambda_1 < 0$, then $D(u)$ can be made negative in M.

Hence, $\lambda_1 \geqq 0$ is the necessary and sufficient condition that $D(u) \geqq 0$ for all curves of the class M and therefore also the necessary and sufficient condition that $D(u) \geqq 0$ for all curves of the class N.

Returning now to the notation of the calculus of variations, we have the theorem:

Theorem XII.—*Suppose* $R > 0$ *and let* λ_1 *denote the smallest characteristic constant of the boundary problem*

$$\frac{d}{dx}\left(R\frac{du}{dx}\right) - (P - Q')u = 0,\ u(x_0) = 0,\ u(x_1) = 0,$$

then $\lambda_1 \geqq 0$ *is the necessary and sufficient condition that*

$$\delta^2 I \geqq 0,\ (\eta).$$

80. Connection with Jacobi's Condition. *a) Sturm's Oscillation Theorem.*—It is *a priori* clear that the condition $\lambda_1 \geqq 0$ must be equivalent to Jacobi's condition.[1] The connection between the two can be established by means of Sturm's oscillation theorem.

Since the differential equation

$$L(u) + \lambda u = 0$$

has no singularities on $\left[ab\right]$, there exists one and only one solution for which

$$u(a) = 0,\ u'(a) = 1.$$

Let us call this $u = V(x, \lambda)$, so that

$$V(a, \lambda) = 0,\ V'(a, \lambda) = 1.$$

Any other solution $u(x)$, for which $u(a) = 0$, is then of the form

$$u = C \cdot V(x, \lambda).$$

Now the boundary problem

$$L(u) + \lambda u = 0,\ u(a) = 0,\ u(b) = 0$$

has for $\lambda = \lambda_1$ a solution $u = \psi_1(x)$, for which

$$\psi_1(a) = 0,\ \psi_1(b) = 0.$$

Hence

$$\psi_1(x) = C \cdot V(x, \lambda_1)$$

and, therefore,

$$V(a, \lambda_1) = 0,\ V(b, \lambda_1) = 0.$$

Let us designate by $\xi(\lambda)$ the root of $V(x, \lambda)$ next greater than a. Then Sturm's oscillation theorem states[2] that

[1] *Cf.* Bolza, "Lectures," §16.

[2] See Bôcher, *Bull.* Am. Math. Soc., 4, 1898.

1) as λ increases, $\xi(\lambda)$ decreases

(49) as λ decreases, $\xi(\lambda)$ increases

2) $V(x, \lambda_1) \neq 0$ between a and b and, therefore, $\xi(\lambda_1) = b$.

b) The Conjugate Point.—Consider the solution

$$u = V(x, 0)$$

and put

$$a' = \xi(0).$$

Case I.—$\lambda_1 > 0$. Then, by (49), as λ decreases from λ_1 to 0, $\xi(\lambda)$ increases from $\xi(\lambda_1) = b$ to $\xi(0) = a'$ and, therefore, $a' > b$.

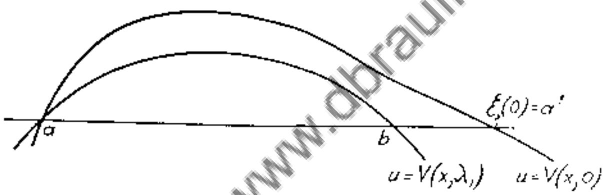


FIG. 19.

Case II.—$\lambda_1 = 0$. Then $a' = b$.

Case III.—$\lambda_1 < 0$. Then, by (49), as λ increases from λ_1 to 0, $\xi(\lambda)$ decreases from $\xi(\lambda_1) = b$ to $\xi(0) = a'$ and, therefore, $a' < b$.

Let us now return to the notation of the calculus of variations, that is

from
$$a,\ b,\ \frac{d}{dx}\left(p\frac{du}{dx}\right) + qu$$
to
$$x_0,\ x_1,\ \frac{d}{dx}\left(R\frac{du}{dx}\right) - (P - Q')u.$$

Then $u = V(x, 0)$ is defined as that integral of Jacobi's differential equation

$$\frac{d}{dx}\left(R\frac{du}{dx}\right) - (P - Q')u = 0$$

which satisfies the initial conditions.

$$u(x_0) = 0, \; u'(x_0) = 1.$$

Hence $V(x, 0)$ is, up to a constant factor, identical with the function denoted in the calculus of variations by $\Delta(x, x_0)$ and, therefore, a' is identical with x_0', the abscissa of the point conjugate to x_0.

This establishes the equivalence between Hilbert's condition

$$\lambda_1 \geqq 0$$

and Jacobi's condition

$$x_1 \leqq x_0'$$

for a non-negative sign of $\delta^2 I$.

III. Vibration Problems

81. Vibrating String. *a) Reduction to Boundary Problem.* For the homogeneous string we had

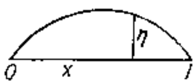


Fig. 20.

(50) $$\frac{\partial^2 \eta}{\partial t^2} = C^2 \frac{\partial^2 \eta}{\partial x^2}$$

(51) $$\eta(0, t) = 0, \; \eta(1, t) = 0, \; (t)$$

(52) $$\eta(x, 0) = f(x), \; \eta_t(x, 0) = F(x), \; (x)$$

where

$$f(0) = 0, \; f(1) = 0$$
$$F(0) = 0, \; F(1) = 0.$$

C was a real positive constant and

$$C^2 = \frac{P}{\kappa\sigma}, \text{ where}$$

P = normal tension, κ = density, σ = area of cross-section.

The differential equation (50) also holds[1] when κ, σ are given functions of x, *i.e.*, the string is non-homogeneous. C^2 is then an always positive, given function of x.

[1] See Weber, "Differentialgleichungen," vol. 2, p. 201.

We try for a solution of the form

$$\eta = u(x)\cdot\phi(t).$$

Substitution of this expression for η in (50) gives

$$u(x)\phi''(t) = C^2u''(x)\phi(t)$$

whence

$$\frac{\phi''(t)}{\phi(t)} = \frac{C^2u''(x)}{u(x)}.$$

But the left-hand member is a function of t alone, the right-hand member is a function of x alone; they are equal, hence equal to the same constant, say $-\lambda$. We are thus led to two ordinary differential equations:

(53) $$\frac{d^2u}{dx^2} + \frac{\lambda}{C^2}u = 0$$

with the boundary conditions

(54) $$u(0) = 0, u(1) = 0,$$

and

(55) $$\frac{d^2\phi}{dt^2} + \lambda\phi = 0.$$

b) The Boundary Problem.—The problem (53) (54) is of the type

(56) $$\begin{cases} L(u) + \lambda gu = 0 \\ R_0(u) = 0,\ R_1(u) = 0, \text{ with } \Big[puu'\Big]_a^b = 0 \end{cases}$$

with $p \equiv 1$, $q \equiv 0$, $g = \frac{1}{C^2} > 0$ on $\Big[ab\Big]$.

We shall show first that every characteristic constant of (56) is positive if

(57) $$p > 0,\ q \leqq 0,\ g > 0.$$

Let λ_0 be a characteristic constant and $\varphi_0(x)$ a corresponding fundamental function of (56):

$$\varphi_0(x) \text{ of class } C'',\ \varphi_0(x) \not\equiv 0,$$
$$L(\varphi_0) + \lambda_0 g\varphi_0 = 0,\ R_0(\varphi_0) = 0,\ R_1(\varphi_0) = 0.$$

Then

$$\lambda_0 \int_a^b g\varphi_0^2 dx = -\int_a^b \varphi_0 L(\varphi_0) dx$$
$$= \int_a^b \left[p\left(\frac{d\varphi_0}{dx}\right)^2 - q\varphi_0^2 \right] dx.$$

Whence, on account of (57),

$$\lambda_0 = \frac{\int_a^b \left[p\left(\frac{d\varphi_0}{dx}\right)^2 - q\varphi_0^2 \right] dx}{\int_a^b g\varphi_0^2 dx} > 0.$$

Hence, under the assumption (57) the condition (H_2) is always satisfied.

Applying this result to the special case (53) (54), we obtain the following lemma:

Lemma I.—For the boundary problem (53) (54), *all of the characteristic constants are positive.*

Green's function $K(x, t)$ is the same as for the homogeneous string, since it depends only on $L(u)$ and not on g. Therefore

$$K(x, t) = \begin{cases} (1 - t)x, & x \leq t \\ (1 - x)t, & x \geq t. \end{cases}$$

The boundary problem (53) (54) is equivalent to

$$u(x) = \lambda \int_0^1 \frac{K(x, t)}{C^2(t)} u(t) dt.$$

In this integral equation the kernel $\dfrac{K(x, t)}{C^2(t)}$ is not symmetric. But, since $C(x) \neq 0$, this integral equation is reducible to one in which the kernel is symmetric by putting

$$(58) \quad u(x) = C(x)\bar{u}(x) \text{ and } K(x, t) = C(x)C(t)\overline{K}(x, t).$$

Then

$$\bar{u}(x) = \lambda \int_0^1 \overline{K}(x, t)\bar{u}(t) dt.$$

By the same transformation (58) the differential equation (53) is transformed into

(59) $$\frac{d}{dx}\left(C^2\frac{d\bar{u}}{dx}\right) + CC''\bar{u} + \lambda\bar{u} = 0$$

with the boundary conditions

(60) $$\bar{u}(0) = 0, \bar{u}(1) = 0$$

and it is easily shown that $K(x, t)$ is Green's function belonging to the boundary problem (59) (60).

We can now apply the Theorems V to $VIII$, with the following result:

The boundary problem (59) (60) has an infinitude of real characteristic constants all of index 1, forming an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

with corresponding normalized fundamental functions

$$\bar{\psi}_1(x), \bar{\psi}_2(x), \bar{\psi}_3(x), \ldots.$$

But, if for a given λ, the boundary problem (59) (60) has a non-trivial solution, then for the same λ, (53) (54) has a non-trivial solution $u = C\bar{u}$. Therefore we have the following lemma:

Lemma II.—The boundary problem (53) (54) *has an infinitude of real positive characteristic constants, all of index* 1, *forming an increasing sequence:*

$$0 < \lambda_1 < \lambda_2 < \ldots,$$

with corresponding normalized fundamental functions

$$\psi_1(x), \psi_2(x), \ldots$$

where

$$\psi_n(x) = C\bar{\psi}_n(x).$$

c) The Generalized Fourier Series.—We return now to (55) with $\lambda = \lambda_n$:

(61) $$\frac{d^2\phi}{dt^2} + \lambda_n\phi = 0.$$

Since $\lambda_n > 0$, the general solution of (61) is

$$\phi = A_n \cos \sqrt{\lambda_n}t + B_n \sin \sqrt{\lambda_n}t.$$

Therefore, presumably a solution of (50) which satisfies (51) is

$$(62) \qquad \eta = \sum_{n=1}^{\infty} (A_n \cos \sqrt{\lambda_n}t + B_n \sin \sqrt{\lambda_n}t)\psi_n(x).$$

In order to be a solution, the condition (52) must be satisfied. Imposing condition (52), we obtain

$$(63) \qquad \begin{cases} \eta(x, 0) = \sum_{n=1}^{\infty} A_n\psi_n(x) = f(x) \\ \eta_t(x, 0) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}B_n\psi_n(x) = F(x). \end{cases}$$

But $f(x)$ and $F(x)$ are given functions. Hence (63) can be satisfied if A_n and B_n can be determined so that the series in (63) represents $f(x)$ and $F(x)$. Since $f(0) = 0$, $f(1) = 0$, $F(0) = 0$, $F(1) = 0$, it follows, from Theorem VI, that $f(x)$ and $F(x)$ can be so represented if f and F are of class C'', and then

$$A_n = (f\psi_n), \quad \sqrt{\lambda_n}B_n = (F\psi_n).$$

We can be sure that (62) satisfies the differential equation if the series is twice differentiable, term by term, with respect to x and t. Since

$$\psi_n''(x) = -\frac{\lambda_n}{C^2}\psi_n(x)$$

this means if

$$\sum \lambda_n(A_n \cos \sqrt{\lambda_n}t + B_n \sin \sqrt{\lambda_n}t)\psi_n(x)$$

is uniformly convergent in x and t.

d) Special Case of Homogeneous String.—If c is constant, then

$$\lambda_n = n^2\pi^2c^2$$

and
$$\psi_n(x) = \sqrt{2} \sin n\pi x,$$

while

$$\eta = \sum_{n=1}^{\infty} (A_n \cos n\pi ct + B_n \sin n\pi ct) \sin n\pi x$$

$$\equiv \sum_{n=1}^{\infty} \eta_n, \text{ say.}$$

For $f(x)$ and $F(x)$ we obtain:

$$f(x) = \sum_{n=1}^{\infty} \sqrt{2} A_n \sin n\pi x$$

$$F(x) = \sum_{n=1}^{\infty} \sqrt{2} n\pi c\, B_n \sin n\pi x.$$

These are sine series for $f(x)$ and $F(x)$. For the development of an arbitrary function in trigonometric series we need know only that the function is continuous and has a finite number of maxima and minima. These conditions are not so strong as those obtained by means of the theory of integral equations which were demanded for the development in series of fundamental functions.

η_1 is periodic in t with period $T_1 = \frac{2}{c}$, this being the period of the fundamental tone. η_n is periodic with period $T_n = \frac{2}{nc} = \frac{T_1}{n}$, and with intensity $\sqrt{A_n^2 + B_n^2}$. Upon the intensity of the different harmonics depends the quality of the tone. The tone of period $\frac{T}{n}$ is called the nth harmonic overtone, or simply the nth harmonic.

For the non-homogeneous string η_n is also periodic with period $\frac{2\pi}{\sqrt{\lambda_n}}$.

T_n decreases with n; the different periods are not fractions of T_1, hence the total motion is not periodic.

82. Vibrations of a Rope. *a) Differential Equation of the Problem.* Let us consider a heavy rope of length 1

$$AB = 1$$

suspended at one end A. It is given a small initial displacement in a vertical plane through AB and then each particle is given an initial velocity. The rope is supposed to vibrate in a given vertical plane and the displacement is so small that each particle is supposed to move horizontally; the cross-section is constant; the density constant; the cross-section infinitesimal compared to the length.

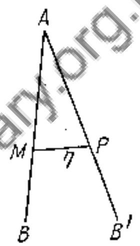


FIG. 21.

Let AB' be the position of the rope at time t and P any point on AB'. Draw PM horizontal and put

$$MP = \eta, \ BM = x.$$

Then the differential equation[1] of the motion is given by

$$(64) \qquad \frac{\partial^2 \eta}{\partial t^2} = C^2 \frac{\partial}{\partial x}\left(x \frac{\partial \eta}{\partial x}\right)$$

where $\qquad C^2 = \text{constant},$

with the boundary conditions

$$(65) \qquad \eta(1, t) = 0, \qquad \eta(0, t) \text{ finite}$$

$$(66) \qquad \eta(x, 0) = f(x), \quad \eta_t(x, 0) = F(x).$$

We try for a solution of the form

$$\eta = u(x) \cdot \phi(t).$$

Substitute this expression for η in (64). We obtain

$$u(x)\phi''(t) = C^2 \phi(t) \frac{d}{dx}\left(x \frac{du}{dx}\right).$$

[1] See KNESER, "Die Integralgleichungen und ihre Anwendung in der Mathematischen Physick," §11.

Whence

$$\frac{\phi''(t)}{\phi(t)} = C^2 \frac{\frac{d}{dx}\left(x\frac{du}{dx}\right)}{u(x)} = -\lambda C^2, \text{ constant.}$$

That is
$$\frac{d^2\phi}{dt^2} + \lambda C^2\phi = 0,$$

and

(67)
$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \lambda u = 0$$

with the boundary condition derived from (65):

(68)
$$u(0) \text{ finite}, u(1) = 0.$$

Equation (67) is of the form

$$L(u) + \lambda u = 0$$

with

$$p = x, q = 0.$$

In the general case $p \neq 0$ on $\left[ab\right]$. This condition is not fulfilled in (67), for $p(0) = 0$. We have also the condition $u(0)$ finite, which did not appear in the general case. The differential equation (67) has a singular point for $x = 0$.

b) Solution of the Boundary Problem.—We solve the boundary problem (67), (68) directly. Put

$$x = \frac{t^2}{4\lambda},$$

then (67) becomes

(69)
$$t\frac{d^2u}{dt^2} + \frac{du}{dt} + tu = 0.$$

We try to find a solution of the form

$$u = \sum_{n=0}^{\infty} C_n t^n.$$

If we substitute this series for u in (69), we obtain

$$C_1 + \sum_{n=1}^{\infty}\left[C_{n-1} + (n+1)^2 C_{n+1}\right] t^n \underset{t}{\equiv} 0.$$

Therefore

$$C_1 = 0,\ C_{n-1} + (n+1)^2 C_{n+1} = 0,\ n = 1, 2, \ldots.$$

That is,

$$C_{n+1} = -\frac{C_{n-1}}{(n+1)^2}.$$

Whence

$$C_1 = C_3 = C_5 = \ldots = C_{2\nu+1} = 0 \ \ldots$$

$$C_2 = -\frac{C_0}{2^2},\ C_4 = \frac{C_0}{2^2 \cdot 4^2},\ C_6 = -\frac{C_0}{2^2 \cdot 4^2 \cdot 6^2},\ \ldots.$$

Therefore

$$u = C_0\left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots\right].$$

By comparison with the series for e^x, which is permanently convergent, we show that this series for u is permanently convergent. This series also satisfies (69). Put

(70) $$J(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

$J(t)$ is called Bessel's function of order zero.

$J(t)$ is then a solution of (69). Knowing a particular solution, we can, by means of Green's formula, find the general solution:

$$t\left[J(t)V' - VJ'(t)\right] = C_1 \neq 0 (C_1 \text{ constant}).$$

Divide both members by $J^2(t)$ and integrate. We find

$$\frac{V}{J(t)} = C_1 \int \frac{dt}{tJ^2(t)} + C_0.$$

But from (70)

$$\frac{1}{J^2(t)} = 1 + t^2 P(t^2),$$

$P(t^2)$ being a power series in t^2, whence

$$\int \frac{dt}{tJ^2(t)} = \log t + P_1(t^2)$$

and

$$V = C_0 J(t) + C_1 J(t)\left\{\log t + P_1(t^2)\right\}$$

where $P_1(t^2)$ is a power series in t^2.

This is the general solution of (69). The solution of (67) is given by putting

$$t = 2\sqrt{\lambda}\sqrt{x}.$$

Whence

$$u = C_0 J(2\sqrt{\lambda x}) + C_1 J(2\sqrt{\lambda x})\left\{\log 2\sqrt{\lambda x} + P_1(4\lambda x)\right\}$$

But, $u(0)$ finite, is a condition upon the solution and this is impossible unless $C_1 = 0$. Therefore

(71) $$u = C_0 J(2\sqrt{\lambda x})$$

is the most general solution of (67), which satisfies the first initial condition. We have the further condition $u(1) = 0$, whence

$$J(2\sqrt{\lambda}) = 0.$$

The solution of this equation gives us the characteristic constants.

c) Construction of Green's Function.—We construct for the boundary problem (67) (68) the Green's function $K(x, t)$ satisfying the following conditions:

A) K continuous on $\left[01\right]$.

B) K of class C'' on $\left[0t\right]\left[t1\right]$ separately.

$$\frac{d}{dx}\left(x\frac{dK}{dx}\right) = 0 \text{ on } \left[0t\right]\left[t1\right] \text{ separately.}$$

C) $K(0, t)$ finite, $K(1) = 0$.

D) $K'(t - 0) - K'(t + 0) = \frac{1}{t}$.

Integrating the differential equation in *B)*, we obtain

$$K(x, t) = \begin{cases} \alpha_0 \log x + \beta_0, & \left[0t\right] \\ \alpha_1 \log x + \beta_1, & \left[t1\right]. \end{cases}$$

But $\qquad K(0, t)$ is finite, therefore $\alpha_0 = 0$,
and $\qquad K(1, t) = 0, \qquad$ therefore $\beta_1 = 0$.
Hence

$$K(x, t) = \begin{cases} K_0(x, t) = \beta_0 \quad , \left[0t\right] \\ K_1(x, t) = \alpha_1 \log x, \left[t1\right]. \end{cases}$$

From condition A)

$$\beta_0 = \alpha_1 \log t.$$

From condition D), since $K'(t - 0) = 0$ and $K'(t + 0) = \frac{\alpha_1}{t}$, we obtain $\alpha_1 = -1$.
Therefore

$$(72) \qquad K(x, t) = \begin{cases} K_0(x, t) = -\log t, \left[0t\right] \\ K_1(x, t) = -\log x, \left[t1\right]. \end{cases}$$

We observe that $K(x, t)$ is symmetric. The graph of $K(x, t)$ for t fixed is shown by the full line in the accompanying figure. As a function of the two variables x and t, $K(x, t)$ is continuous on $\left[01\right]$ except at $x = t = 0$. For in the region marked I we have $t > x$ and

$$K(x, t) = \log \frac{1}{t},$$

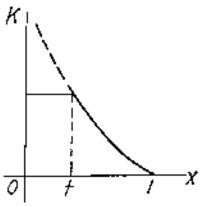


FIG. 22.

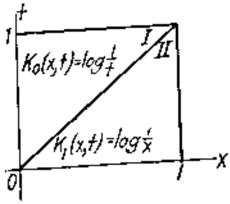


FIG. 23.

while in the region marked II we have $t < x$ and

$$K(x, t) = \log \frac{1}{x}.$$

For $x = 0$, $t > 0$, we have $K = -\log t$, finite.
For $t = 0$, $x > 0$, we have $K = -\log x$, finite.

d) Equivalence with a Homogeneous Integral Equation.—We have

$$L(u) = -\lambda u$$
$$L(K) = 0.$$

Multiply the first of these by $-K$ and the second by u and add. We obtain

$$uL(K) - KL(u) = \lambda uK.$$

Integrate both members of this expression from 0 to $t - 0$ and from $t + 0$ to 1 with respect to t and add. We obtain

$$u(x) = \lambda \int_0^1 K(x, t)u(t)dt.$$

The details of the integration are the same as those given several times previously, except for the term

$$x\left[uK' - Ku'\right] \text{ for } x = 0.$$

But here u and K are finite by hypothesis. From (71) we see that if u is finite so also is u'. From the explicit expression (72) for K we find $K'(0, t) \equiv 0$. Therefore

$$x\left[uK' - K'u\right] = 0 \text{ for } x = 0.$$

The kernel

$$K(x, t) = \begin{cases} -\log t, & \left[0t\right] \\ -\log x, & \left[t1\right] \end{cases}$$

is symmetric, but it is discontinuous at one point, *viz.*, $x = t = 0$. Schmidt (page 21 of his dissertation) has shown that the results of the Hilbert-Schmidt theory of continuous symmetric kernels still hold for a discontinuous kernel if

1) $\int_a^b K(x, t)f(t)dt$ for f continuous, is continuous in x on $\left[ab\right]$.

2) The iterated kernel $K_2(x, t)$ is continuous and does not vanish identically.

These conditions are satisfied in the present instance and thus all of the Hilbert-Schmidt theory, as well as the Theorems V to VIII on boundary problems, remain true. Therefore we know that there exists an infinite sequence of positive characteristic constants

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdot \cdot \cdot ,$$

with a corresponding complete normalized orthogonal system of fundamental functions. Therefore

$$J(2\sqrt{\lambda}) = 0$$

has an infinitude of positive roots λ_n. Put

$$2\sqrt{\lambda} = k.$$

Then $J(k) = 0$, and the roots are

$$k_n = 2\sqrt{\lambda_n}.$$

The first four values of k_n are[1]

$$k_1 = 2.405,\ k_2 = 5.520,\ k_3 = 8.654,\ k_4 = 11.792$$

and generally $(n - \frac{1}{2})\pi < k_n < n\pi$. Therefore

$$\varphi_n(x) = C_0 J(2\sqrt{\lambda_n x}) = C_0 J(k_n\sqrt{x}).$$

These fundamental functions $\varphi_n(x)$ will become orthogonalized if we choose

$$C_0 = \frac{1}{\sqrt{\int_0^1 J^2(k_n\sqrt{x})dx}}$$

[1] See Fricke, "Analytische-Funktionentheoretische Vorlesungen," p. 74.

But[1]
$$\int_0^1 J^2(k_n\sqrt{x})dx = \frac{1}{[J'(k_n)]^2}.$$

Therefore

$$\psi_n(x) = \frac{J(k_n\sqrt{x})}{J'(k_n)}, \text{ and}$$

$$\eta = \sum_{n=1}^{\infty}\left(A_n \cos \frac{Ck_n}{2}t + B_n \sin \frac{Ck_n}{2}t\right)\psi_n(x).$$

This expression for η satisfies (64) and (65). We now determine A_n, B_n, if possible, in order that (66) may be satisfied. This gives us the two equations

$$\sum A_n\psi_n(x) = f(x)$$
$$\sum \frac{Ck_n}{2}B_n\psi_n(x) = F(x).$$

Since
$$f(0) \text{ is finite, } f(1) = 0,$$
$$F(0) \text{ is finite, } F(1) = 0,$$

$f(x)$ and $F(x)$ can be expanded as series in $\psi_n(x)$, provided f and F are of class C'', and then

$$A_n = \int_0^1 f(x)\psi_n(x)dx = \int_0^1 f(x)\frac{J(k_n\sqrt{x})}{J'(k_n)}dx,$$

while
$$\frac{Ck_n}{2}B_n = \int_0^1 F(x)\psi_n(x)dx.$$

83. The Rotating Rope.—(See KNESER, page 46.)

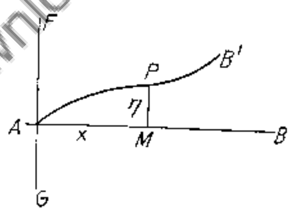


FIG. 24.

a) The Problem and Its Differential Equation.—Let FG be an axis around which a plane is rotating with constant velocity; a rope AB is attached at a point A of the axis and constrained to remain in the rotating plane. The velocity of the rotation is so large that the weight of the rope can be neglected. Then the straight line AB perpendicular to FG

[1] See FRICKE. *Loc. cit.*, p. 65.

is a relative position of equilibrium for the rope. Displace the rope slightly from this position AB, then let it go after imparting to its particles initial velocities perpendicular to AB. The rope will then describe small vibrations around the position of equilibrium.

Let APB' be the position of the rope at the time t, P one of its points, $PM \perp AB$. Put $AM = x$, $MP = \eta$, and suppose $AB = 1$. Then the function $\eta(x, t)$ must satisfy the partial differential equation

$$\frac{\partial^2\eta}{\partial t^2} = C^2\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial\eta}{\partial x}\right],\ C \text{ constant},$$

the boundary conditions

$$\eta(0, t) = 0,\ \eta(1, t) = \text{finite},\ (t),$$

and the initial conditions

$$\eta(x, 0) = f(x),\ \eta_t(x, 0) = F(x),\ (x).$$

Putting
$$\eta = u(x)\,\phi(t)$$
we obtain

$$(73)\qquad \frac{d}{dx}\left[(1 - x^2)\frac{du}{dx}\right] + \lambda u = 0$$
$$u(0) = 0,\ u(1) = \text{finite}$$
$$\frac{d^2\phi}{dt^2} + \lambda C^2 u = 0.$$

b) Solution of the Boundary Problem.—$x = 1$ is a singular point of (73). By Fuch's theory[1] (73) has for every λ one and only one solution, which in the vicinity of $x = 1$ is given by

$$u_1 = 1 + a_1(x - 1) + a_2(x - 1)^2 + \cdots$$

Every other solution u_2 is obtained from Green's formula

$$(1 - x^2)(u_1u_2' - u_2u_1') = C_1 \neq 0.$$

[1] See Goursat, "Cours D'Analyse," tome 2, §412.

Therefore

$$\frac{d}{dx}\frac{u_2}{u_1} = \frac{C_1}{(x^2 - 1)u_1^2} = C_1\left\{\frac{1}{x - 1} + S_1(x - 1)\right\}.$$

Whence

$$u_2 = C_0 u_1 + C_1 u_1 \left\{\log (x - 1) + S(x - 1)\right\}.$$

Therefore the condition, $u(1)$ finite, leads to $C_1 = 0$ and hence

$$u = C_0 u_1 \tag{74}$$

is the most general solution of (73) which satisfies the second boundary condition. Put

$$u_1 = U(x, \lambda).$$

Then the first boundary condition gives

$$U(0, \lambda) = 0$$

for the determination of the characteristic constants. From (74) we see that each characteristic constant is of index 1.

Let λ_0 be a characteristic constant and $\varphi_0(x) = U(x, \lambda_0)$ a corresponding fundamental function. Then $\varphi_0(x)$ is an odd function:

$$\varphi_0(-x) = -\varphi(x).$$

For u_1 can be expanded according to powers of x, say

$$u_1 = \sum_{r=1}^{\infty} C_r x^r.$$

The substitution of this series in the differential equation leads to the following recurrent formula

$$C_{r+2} = \frac{r(r + 1) - \lambda_0}{(r + 1)(r + 2)} C_r, \; r = 0, 1, 2, \; . \; . \; . \; ,$$

which shows that if $C_0 = 0$, then

$$C_{2\mu} = 0, \; \mu = 0, 1, 2, \; . \; . \; . \; .$$

But from $U(0, \lambda) = 0$, we obtain $\varphi_0(0) = 0$, hence $C_0 = 0$, and, therefore, $\varphi_0(x)$ is an odd function.

We can at once indicate some of the characteristic constants and fundamental functions from the theory of Legendre's polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2 - 1)^n\right]$$
$$P_0(x) = 1.$$

$P_n(x)$ is a rational integral function of degree n, satisfying the differential equation

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + n(n + 1)y = 0.$$

Furthermore, $P_{2n}(x)$ is an even function and $P_{2n}(0) \neq 0$, $P_{2n-1}(x)$ is an odd function and $P_{2n-1}(0) = 0$. Therefore

$$\lambda_n = 2n(2n - 1),\ n = 1, 2, 3, \ldots$$

is a characteristic constant and $P_{2n-1}(x)$ a corresponding fundamental function.

The characteristic constants are of index 1 since the condition $u(1) = 0$ determines u up to a constant factor.

We will now show that there are no other characteristic constants than

$$\lambda_n = 2n(2n - 1),\ n = 1, 2, 3, \ldots$$

Suppose λ_0 to be a characteristic constant and $\lambda_0 \neq \lambda_n$ and $\psi_0(x)$ a corresponding normalized fundamental function, then, according to the orthogonality theorem (§59),

$$\int_0^1 \psi_0(x)P_{2n-1}(x)dx = 0.$$

Now $\psi_0(x)$ is continuous on the interval $\left[-1, +1\right]$ and, therefore, by a theorem due to Weierstrass,[1] can be expanded in an infinite series of polynomials uniformly convergent on $\left[-1, +1\right]$:

[1] GOURSAT-HEDRICK, "Mathematical Analysis," vol. 1, §199.

(75) $$\psi_0(x) = \sum_{r=1}^{\infty} G_r(x).$$

But, as shown under b),

$$\psi_0(-x) = -\psi_0(x).$$

Therefore, from

$$\psi_0(-x) = \sum_{r=1}^{\infty} G_r(-x)$$

and (75) above, we obtain

$$\psi_0(x) = \frac{1}{2}\sum_{r=1}^{\infty}\left[G_r(x) - G_r(-x)\right] \equiv \sum_{r=1}^{\infty} H_r(x)$$

and $\sum H_r(x)$ is uniformly convergent on $\left[-1, +1\right]$. From the definition of $H_r(x)$ we obtain

$$H_r(-x) = -H_r(x),$$

that is, $H_r(x)$ is an odd function.

From the uniform convergence of $\sum H_r(x)$ we have that for every positive ϵ it is possible to find an m such that

(76) $$\left|\psi_0(x) - \sum_{r=1}^{m} H_r(x)\right| < \epsilon, \left[-1, +1\right].$$

Let us choose

(77) $$0 < \epsilon < \frac{1}{\int_0^1 |\psi_0(x)| dx}.$$

Since $H_r(x)$ is an odd function we have

(78) $$\sum_{r=1}^{m} H_r(x) = \sum_{r=0}^{n} C_r x^{2r-1}.$$

But $P_{2n-1}(x)$ are odd functions:

$$P_1(x) = a_{11}x$$
$$P_3(x) = a_{22}x^3 + a_{21}x$$
$$\cdots\cdots\cdots\cdots$$
$$P_{2n-1}(x) = a_{nn}x^{2n-1} + \cdots + a_{n1}x$$

with $$a_{11}, a_{22}, \ldots, a_{nn} \neq 0.$$

These equations can be solved sequentially for $x, x^3, \ldots, x^{2n-1}$ in terms of $P_1, P_3, \ldots, P_{2n-1}$. Put these values of x^{2r-1} in (78) above and we will have

$$\sum_{r=1}^{m} H_r(x) = \sum_{r=0}^{n} C_r P_{2r-1}(x).$$

This expression for $\sum H_r(x)$, substituted in (76), gives

$$\left| \psi_0(x) - \sum_{r=0}^{n} C_r P_{2r-1}(x) \right| < \epsilon.$$

Put

(79) $$\psi_0(x) = \sum_{r=0}^{n} C_r P_{2r-1}(x) + r(x),$$

whence

$$|r(x)| < \epsilon \text{ on } \left[-1, +1 \right].$$

From (79) we obtain

$$\int_0^1 \psi_0^2(x)dx = \sum_{r=0}^{n} C_r \int_0^1 \psi_0(x)P_{2r-1}(x)dx + \int_0^1 r(x)\psi_0(x)dx.$$

But $$\int_0^1 \psi_0(x)P_{2r-1}(x)dx = 0$$

and $$\int_0^1 \psi_0^2(x)dx = 1.$$

Therefore $$1 = \int_0^1 r(x)\psi_0(x)dx.$$

Now $\left|\int_0^1 r(x)\psi_0(x)dx\right| \leqq \int_0^1 \left|r(x)\psi_0(x)\right| dx$

$$< \epsilon \int_0^1 \left|\psi_0(x)\right| dx.$$

Therefore
$$1 < \epsilon \int_0^1 \left|\psi_0(x)\right| dx,$$

which, on account of our choice (77) of ϵ, gives

$$1 < \epsilon \int_0^1 \left|\psi_0(x)\right| dx < 1,$$

which constitutes a contradiction. Therefore

$$\lambda_n = 2n\ (2n - 1)$$

are the only characteristic constants, and the only fundamental functions are

$$\varphi_n(x) = \frac{P_{2n-1}(x)}{\sqrt{\int_0^1 P^2{}_{2n-1}(x)dx}}.$$

But
$$\int_0^1 P^2{}_{2n-1}(x)dx = \frac{1}{4n - 1},$$

as shown in the theory of Legendre's polynomials. Therefore

$$\varphi_n(x) = \sqrt{4n - 1}\, P_{2n-1}(x).$$

C) *Equivalence with Integral Equation.*—We construct the Green's function as before and obtain

$$K(x, t) = \begin{cases} \frac{1}{2}\log\frac{1+x}{1-x},\ 0 \leqq x \leqq t \\ \frac{1}{2}\log\frac{1+t}{1-t},\ t \leqq x \leqq 1. \end{cases}$$

$K(x, t)$ is symmetric and has but one point of discontinuity and that is for $x = t = 1$.

Schmidt's conditions for a discontinuous kernel hold, however, and so the theorems of the Hilbert-Schmidt theory apply. Proceeding as in the previous problems, we find that the boundary problem is equivalent to the following integral equation:

$$u(x) = \lambda \int_0^1 K(x, t)u(t)dt.$$

The Theorems V to IX hold for this boundary problem as well as the orthogonality theorem, §59. Hence

$$\int_0^1 P_{2n-1}(x)P_{2m-1}(x)dx = 0, \; m \neq n$$

This last result agrees with a result of the theory of Legendre polynomials. Applying Theorem VI, we see that if f is of class C'' and $f(0) = 0$, then

$$f(x) = \sum_{r=0}^{\infty} C_r P_{2n-1}(x)$$

where $$C_r = (fP_{2n-1})\frac{4n-1}{1}.$$

IV. Applications of the Hilbert-Schmidt Theory to the Flow of Heat in a Bar

84. The Partial Differential Equations of the Problem.[1] *a) General Hypotheses.*—The theory of the flow of heat in a bar is based upon the following hypotheses:

A) Let dm denote the mass of an element of a conductor of temperature θ, C its specific heat, then the amount of heat dQ necessary to increase the temperature of the element from θ to $\theta + d\theta$ is given by the formula

$$dQ = Cdmd\theta.$$

B) Let $d\omega$ be the area of an element of surface through an interior point P of a conductor, n one of the two normals to the element, k the inner conductivity at point P, then the amount of heat which flows through the inner surface $d\omega$ in time dt is

$$dQ = -k\frac{\partial\theta}{\partial n}d\omega dt.$$

[1] See Weber, "Partielle Differentialgleichungen," §§32–34.

C) Let $d\omega$ be an element of the outer surface of the conductor, h the outer conductivity at P, θ the temperature of the conductor at P, Θ the temperature of the surrounding medium at P, then the amount of heat which flows through the outer surface $d\omega$ in time dt is

$$dQ = h(\theta - \Theta)d\omega dt.$$

b) We now apply these principles to the determination of the flow of heat in a straight bar placed along the x-axis of a rectangular system of coordinates, with a cross-section σ, infinitesimal compared to the length, so that the temperature θ may be considered as constant; accordingly, θ will be a function of x and t: $\theta(x, t)$. The bar is imbedded in a medium of given temperature which is also supposed to be a function of x and t: $\Theta(x, t)$. The bar has a given initial temperature $\theta(x, 0) = f(x)$. Consider an element of the bar of length dx; we may regard it as a cylinder.

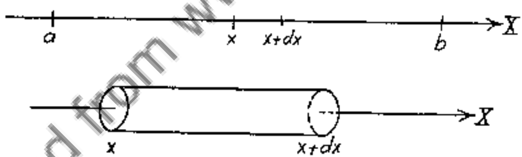


FIG. 25.

1) By B) the quantity of heat which enters at the left-hand end is

$$-k\frac{\partial\theta}{\partial x}\sigma\bigg]_x dt.$$

2) Similarly, the quantity of heat which flows out through the right-hand end is $-k\sigma\dfrac{\partial\theta}{\partial x}\bigg]_{x+dx} dt,$

or, if we expand according to powers of dx and neglect higher powers of dx,

$$-k\sigma\frac{\partial\theta}{\partial x}\bigg]_x dt - \frac{\partial}{\partial x}\left(k\sigma\frac{\partial\theta}{\partial x}\right)\bigg]_x dtdx.$$

3) The quantity of heat which flows out through the cylindrical surface is, by C),

$$h(\theta - \Theta) l dx dt,$$

where l is the periphery of the cross-section. The total amount of heat which enters the element of the bar in the time dt is, therefore,

$$\left\{\frac{\partial}{\partial x}\left(k\,\sigma\,\frac{\partial \theta}{\partial x}\right) - h(\theta - \Theta)l\right\} dx dt.$$

This amount of heat increases the temperature of the element in the time dt by $\frac{\partial \theta}{\partial t}\,dt$ and, therefore, by A), is equal to $C\rho\sigma dx\,\frac{\partial \theta}{\partial t}\,dt$, where ρ is the density. Hence we obtain the partial differential equation

$$C\rho\sigma\,\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k\sigma\,\frac{\partial \theta}{\partial x}\right) - h(\theta - \Theta)l. \tag{80}$$

If, moreover, heat is produced in the interior of the bar by electric currents or other sources of energy, let

$$A(x) dx dt$$

be the quantity of heat produced in the time dt in the interior of the element of the bar between the cross-sections, at x and $x + dx$. Then the partial differential equation (80) becomes

$$c\rho\sigma\,\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(k\sigma\,\frac{\partial \theta}{\partial x}\right) - h(\theta - \Theta)l + A(x).$$

In addition, the temperature θ has to satisfy certain boundary conditions, which are obtained as follows:

The amount of heat which leaves at a is, by C),

$$h(\theta - \Theta)\sigma\Big]_a dt. \tag{81}$$

On the other hand, the quantity of heat which flows through the cross-section $a + h$ in the direction of the negative x-axis in the time dt is, by B),

$$k\sigma \frac{\partial \theta}{\partial x}\bigg]_{a+h} dt.$$

The limit of this expression as h approaches zero must be equal to the quantity (81). Hence we obtain the first boundary condition

$$h(\theta - \Theta) - k \frac{\partial \theta}{\partial x}\bigg]_a = 0.$$

The same reasoning applied to the cross-section b gives us the second boundary condition

$$h(\theta - \Theta) + k \frac{\partial \theta}{\partial x}\bigg]_b = 0.$$

c) The Special Case $\Theta \equiv 0$, $A \equiv 0$.—We now consider the differential equation under the assumption that

$$\Theta \equiv 0, A \equiv 0.$$

The differential equation is, then,

$$c\sigma\rho \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k\sigma \frac{\partial \theta}{\partial x}\right) - hl\theta.$$

Put $\quad c\sigma\rho = g(x),\ k\sigma = p(x),\ hl = -q(x),$

then $\quad g(x) > 0,\ p(x) > 0,\ q(x) \leqq 0,$

and $\quad q(x) = 0$ only when $h = 0$.

The differential equation of the problem now becomes

$$g \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(p \frac{\partial \theta}{\partial x}\right) + q\theta. \tag{82}$$

Put $$\frac{h(a)}{k(a)} = H_0 \geqq 0,\ \frac{h(b)}{k(b)} = H_1 \geqq 0,$$

then the boundary conditions become

$$\begin{cases} \theta_x(a, t) - H_0\theta(a, t) = 0 \\ \theta_x(b, t) + H_1\theta(b, t) = 0 \end{cases} (t). \tag{83}$$

The initial condition is

(84) $$\theta(x, 0) = f(x).$$

$f(x)$ is not entirely arbitrary, for from (83) we obtain for $t = 0$

$$\theta_x(a, 0) - H_0\theta(a, 0) = 0$$
$$\theta_x(b, 0) + H_1\theta(b, 0) = 0$$

whence

(85) $$f'(a) - H_0 f(a) = 0$$
$$f'(b) + H_1 f(b) = 0.$$

To solve (82), put

$$\theta = u(x)\phi(t).$$

Then, in the usual manner, (82) breaks up into the two ordinary differential equations:

(86) $$\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu + \lambda gu = 0$$

(87) $$\frac{d\phi}{dt} + \lambda\phi = 0,$$

while from (83) we get

(88) $$u'(a) - H_0 u(a) = 0$$
$$u'(b) + H_1 u(b) = 0$$

as boundary conditions on (86).

We have, as in our hypothesis (H_1), §72, that p is of class C', $p > 0$. In addition, we have $g > 0$ and $q \leqq 0$.

We shall now show that all of the characteristic constants of (86) are $\geqq 0$. This proof is analagous to that given in the proof of Theorem VIII.

We suppose that λ_0 is a characteristic constant and φ_0 a corresponding fundamental function. Therefore

(89) $$L(\varphi_0) + \lambda_0\varphi_0 g = 0$$

(90) $$\varphi_0'(a) - H_0\varphi_0(a) = 0, \varphi_0'(b) + H_1\varphi_0(b) = 0$$

and $\varphi_0 \not\equiv 0$ is of class C''.

From (89) we obtain

$$\lambda_0 \int_a^b g\varphi_0^2 dx = -\int_a^b \varphi_0 L(\varphi_0) dx.$$

Integration by parts gives

$$\lambda_0 \int_a^b g\varphi_0^2 dx = -p(b)\varphi_0(b)\varphi_0'(b) + p(a)\varphi_0(a)\varphi_0'(a) + \int_a^b (p\varphi_0'^2 - q\varphi_0^2)dx,$$

which, by (90), reduces to

$$\text{(91)} \quad \lambda_0 \int_a^b g\varphi_0^2 dx = p(b)\varphi_0^2(b)H_1 + p(a)\varphi_0^2(a)H_0 + \int_a^b (p\varphi_0'^2 - q\varphi_0^2)dx.$$

Now $p(b)$, $\varphi_0^2(b)$, $p(a)$, $\varphi_0^2(a)$, and p are positive and not zero, and H_1, H_0, $\varphi_0'^2$, φ_0^2, $-q$ are $\geqq 0$. Whence, from (91), we conclude

$$\lambda_0 \geqq 0.$$

The equality will hold only when $H_1 = 0$, $H_0 = 0$, $q \equiv 0$ simultaneously. But $-q = hl$ and $l > 0$. Therefore $h \equiv 0$, which means that no heat escapes through the cylindrical surface. It also follows that no heat escapes through the ends for $h(a) = 0$, $h(b) = 0$, and hence the equality holds only when no heat escapes to the surrounding medium.

We can now construct the Green's function $K(x, t)$ and establish the equivalence of the boundary problem with the integral equation

$$u(x) = \lambda \int_a^b K(x, t)g(t)u(t)dt.$$

The kernel is not symmetric but the substitution

$$u(x)\sqrt{g(x)} = \bar{u}(x), \qquad K(x, t)\sqrt{g(x)g(t)} = \bar{K}(x, t)$$

transforms the problem into one with a symmetric kernel. This transformation leads the given differential equation into one in which $g \equiv 1$. We can then apply the results of the general Hilbert-Schmidt theory. Further, we can apply the Theorems V to VIII. We know then the existence of an infinitude of real characteristic constants each of index 1:

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

with corresponding normalized fundamental functions

$$\psi_1(x), \psi_2(x), \psi_3(x), \cdots$$

We now turn to the solution of (87) for $\lambda = \lambda_n$. The solution is

$$\phi(t) = C_n e^{-\lambda_n t}$$

whence

$$\theta = C_n e^{-\lambda_n t} \psi_n(x)$$

is a solution of (82) which satisfies the boundary conditions but in general will not satisfy the initial conditions (84). Construct

$$\theta = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} \psi_n(x). \tag{92}$$

If we assume that (92) is convergent and admits one term-by-term partial differentiation with respect to t and two term-by-term partial differentiations with respect to x, then (92) satisfies (82) and (83), and (84) will be satisfied if

$$\sum_{n=1}^{\infty} C_n \psi_n(x) = f(x),$$

that is, if $f(x)$ can be expanded into a series of fundamental functions. That this is possible follows from Theorem VI when we take account of (85), provided f is of class C'', and C_n will be given by

$$C_n = (f\psi_n).$$

85. Application to an Example.—We consider now a special case where C, σ, ρ, k, h, l are constants and $a = 0$,

$b = 1$. Hence, p, q, g in (86) are constant. Put $\frac{q}{p} = -b^2$, and $\frac{\lambda g}{p}$ = constant, which we use as a new parameter and again designate by λ. Then (86) becomes

$$\frac{d^2u}{dx^2} + (\lambda - b^2)u = 0.$$

If we now assume that no heat escapes through the two end surfaces, then $H_0 = H_1 = 0$ and the boundary conditions become

$$u'(0) = 0, u'(1) = 0.$$

For this boundary problem the characteristic constants are easily found to be

$$\lambda_0 = b^2, \lambda_n = n^2\pi^2 + b^2, n = 1, 2, 3, \ldots$$

and the normalized fundamental functions are $\psi_0(x) = 1$, $\psi_n(x) = \sqrt{2} \cos n\pi x$.

For the further discussion we have to distinguish two cases:

Case I.—$b \neq 0$. Then all of the characteristic constants are positive. Therefore $\lambda = 0$ is not a characteristic constant and condition (H_2) is satisfied. Hence for this boundary problem we can construct a Green's function satisfying the following conditions:

A) K is continuous on $\left[01\right]$.

B) K is of class C'' on $\left[0t\right] \left[t1\right]$ separately.

$$\frac{d^2K}{dx^2} - b^2K = 0 \text{ on } \left[0t\right] \left[t1\right] \text{ separately.}$$

C) $K'(0) = 0, K'(1) = 0$.

D) $K'(t - 0) - K'(t + 0) = 1$.

The Green's function satisfying these conditions is easily found to be

$$K(x, t) = \begin{cases} \dfrac{\cosh bx \cosh b(1 - t)}{b \sinh b}, & 0 \leq x \leq t \\ \dfrac{\cosh bt \cosh b(1 - x)}{b \sinh b}, & t \leq x \leq 1. \end{cases}$$

It is symmetric and the boundary problem is equivalent to the integral equation

$$u(x) = \lambda \int_0^1 K(x, t) u(t) dt.$$

Theorem VI of this chapter assures us that every fundamental function of class C'' of this boundary problem, for which $u'(0) = 0$, $u'(1) = 0$ can be expanded into a cosine series

$$u(x) = \sum_{n=0}^{\infty} C_n \cos n\pi x$$

convergent on the interval $0 \leq x \leq 1$.

Case II.—$b = 0$. The boundary problem now becomes

$$\frac{d^2u}{dx^2} + \lambda u = 0, \; u'(0) = 0, \; u'(1) = 0, \tag{93}$$

for which the characteristic constants are

$$\lambda_0 = 0, \; \lambda_n = n^2\pi^2, \; n = 1, 2, 3, \; . \; . \; .$$

with the corresponding normalized fundamental functions

$$\psi_0(x) = 1, \; \psi_n(x) = \sqrt{2} \cos n\pi x.$$

Let us consider

$$\bar{K}(x, t) = K(x, t) - \frac{1}{b^2}.$$

From the conditions on K we find that $\bar{K}$ satisfies the same conditions. Now let

$$\lim_{b \to 0} \bar{K}(x, t) = H(x, t).$$

We find

$$H(x, t) = \begin{cases} \frac{x^2 + t^2}{2} + \frac{1}{3} - t \equiv H_0, & 0 \leq x \leq t \\ \frac{x^2 + t^2}{2} + \frac{1}{3} - x \equiv H_1, & t \leq x \leq 1. \end{cases}$$

We see that $H(x, t)$ is symmetric and satisfies the following conditions:

A) H is continuous on $\left[01\right]$.

B) H is of class C'' on $\left[0t\right] \left[t1\right]$ separately.

$$\frac{d^2H}{dx^2} = 1 \text{ on } \left[0t\right] \left[t1\right] \text{ separately.}$$

C) $H'(0) = 0,\ H'(1) = 0.$

D) $H'(t - 0) - H'(t + 0) = 1.$

E) $\int_0^1 H(x, t)dt = 0,\ (t).$

We will now show that the boundary problem (93) is equivalent to

$$u(x) = \lambda \int_0^1 H(x, t)u(t)dt.$$

A) Multiply both members of $\frac{d^2u}{dx^2} = -\lambda u$ by $-H$, and both members of $\frac{d^2H}{dx^2} = 1$ by u and add. We obtain

$$\frac{d}{dx}(uH' - Hu') = \lambda uH + u,$$

whence

$$(uH' - Hu')\Big]_0^{t-0} + (uH' - Hu')\Big]_{t+0}^1 = \int_0^1 u(x)dx + \lambda \int_0^1 H(x, t)u(x)dx,$$

which, on account of (93) and condition C) on H, reduces to

$$(uH' - Hu')\Big]_{t+0}^{t-0} = \lambda\int_0^1 u(x)dx + \lambda\int_0^1 H(x, t)u(x)dx,$$

which, by conditions A) and D) on H, reduces to

$$u(t) = \lambda\int_0^1 u(x)dx + \lambda\int_0^1 H(x, t)u(x)dx.$$

Now from

$$\frac{d^2u}{dx^2} = -\lambda u(x)$$

we obtain, by integration,

$$-\int_0^1 u(x)dx = \int_0^1 \frac{d^2u}{dx^2}dx = \frac{du}{dx}\Big]_0^1 = 0.$$

Therefore

$$(94) \qquad u(x) = \lambda\int_0^1 H(x, t)u(t)dt.$$

Thus we have shown that every solution of the boundary problem (93) with $\lambda \neq 0$ is a solution of the integral equation (94).

B) Conversely, suppose that u is continuous and

$$(95) \qquad u(x) = \lambda\int_0^1 H(x, t)u(t)dt, \text{ and } \lambda \neq 0.$$

Then

$$u(x) = \lambda\int_0^x H_1(x, t)u(t)dt + \lambda\int_x^1 H_0(x, t)u(t)dt,$$

whence

$$(96) \quad u'(x) = \lambda\int_0^x H_1'(x, t)u(t)dt + \lambda\int_x^1 H_0'(x, t)u(t)dt$$

and

$$u''(x) = \lambda\int_0^x H_1''(x, t)u(t)dt + \lambda\int_x^1 H_0''(x, t)u(t)dt + \lambda H_1'(x, x)u(x) - \lambda H_0'(x, x)u(x).$$

But $\qquad H'' = 1$ and $H'(t - 0) - H'(t + 0) = 1.$

Therefore

$$u''(x) + \lambda u(x) = \lambda\int_0^1 u(t)dt.$$

But from (95)

$$\int_0^1 u(x)dx = \lambda \int_0^1 \int_0^1 H(x, t)u(t)dtdx$$
$$= \lambda \int_0^1 u(t)dt \int_0^1 H(x, t)dx = 0$$

on account of E). Therefore

$$u''(x) + \lambda u(x) = 0.$$

Further, from (96) we obtain

$$u'(0) = \lambda \int_0^1 H_0'(0, t)u(t)dt = 0$$
$$u'(1) = \lambda \int_0^1 H_1'(1, t)u(t)dt = 0,$$

on account of C). Therefore we have proved the equivalence of the boundary problem and integral equation.

86. General Theory of the Exceptional Case.—The method just applied to the special boundary problem (93) is a special case of the general method which Hilbert uses in the exceptional case where $\lambda = 0$ is a characteristic constant of the boundary problem:

(97) $$L(u) + \lambda u = 0$$
(98) $$R_0(u) = 0, \; R_1(u) = 0.$$

Let $\psi_0(x)$ be a normalized fundamental function belonging to $\lambda = 0$, so that ψ_0 is of class C'',

$$L(\psi_0) = 0, \; R_0(\psi_0) = 0, \; R_1(\psi_0) = 0,$$
$$\int_a^b \psi_0^2(x)dx = 1.$$

Then, according to Hilbert,[1] the equivalence of (97) (98) with an integral equation can be established as follows.

a) The Modified Green's Function.—

[1] Gött. Nach., 2, p. 219, 1904.

Theorem I.—*Under the above assumptions there exists one and only one function $H(x, t)$ satisfying as a function of x the following conditions:*

A) H is continuous on $\left[ab\right]$.

B) H is of class C'' on $\left[at\right]\left[tb\right]$ separately.

$$L(H) = \psi_0(x)\psi_0(t) \text{ on } \left[at\right]\left[tb\right] \text{ separately.}$$

C) $R_0(H) = 0$, $R_1(H) = 0$.

D) $H'(t - 0) - H'(t + 0) = \dfrac{1}{p(t)}$.

E) $\displaystyle\int_a^b H(x, t)\psi_0(x)dx = 0, (t).$

$H(x, t)$ is called the modified Green's function. An outline of the proof of this theorem is as follows:

Let $u_0(x)$ be a particular solution of

$$L(u) = \psi_0(x)\psi_0(t),$$

then the general solution is

$$u = u_0(x) + \alpha\psi_0(x) + \beta V(x),$$

where V is a particular solution of $L(u) = 0$, independent of $\psi_0(x)$.

Now, by Green's theorem,

$$p(\psi_0 V' - \psi_0' V) = C$$

and we can select V in such a way that $C = 1$.

We now put

$$H(x, t) = \begin{cases} u_0(x) + \alpha_0\psi_0(x) + \beta_0 V(x), & x \leq t \\ u_0(x) + \alpha_1\psi_0(x) + \beta_1 V(x), & t \leq x. \end{cases}$$

Then

$$(99) \qquad \begin{aligned} R_0(H) &= R_0(u_0) + \beta_0 R_0(V) = 0 \\ R_1(H) &= R_1(u_0) + \beta_1 R_1(V) = 0 \end{aligned}$$

determine β_0, β_1, since $R_0(V) \neq 0$, $R_1(V) \neq 0$.

The conditions A) to D) determine $\alpha_0 - \alpha_1$ and $\beta_0 - \beta_1$:

$$\alpha_0 - \alpha_1 = -V(t),\ \beta_0 - \beta_1 = \psi_0(t).$$

Since β_0, β_1 have already been determined, this furnishes an apparent contradiction, but it will be found that the determination of $\beta_0 - \beta_1$ is a consequence of the determination of β_0 and β_1 separately. Thus the conditions A) to D) determine $\beta_0, \beta_1, \alpha_0, \alpha_1$, except for an additive constant which is determined by E).

Theorem II. *The function $H(x, t)$ is symmetric.*—For let

$$a \leqq t \leqq s \leqq b.$$

Multiply each member of

$$L\left\{H(x, t)\right\} = \psi_0(x)\psi_0(t)$$

by $-H(x, s)$ and each member of

$$L\left\{H(x, s)\right\} = \psi_0(x)\psi_0(s)$$

by $H(x, t)$. Add and integrate from a to t, t to s, s to b and add the results. We obtain

$$H(t, s) = H(s, t).$$

b) Hilbert's fundamental Theorem III now takes the form

Theorem III.—*If f is continuous, then the statements F is of class C'', $L(F) + f = 0$, $R_0(F) = 0$, $R_1(F) = 0$ imply and are implied by*

$$F(x) = \int_a^b H(x, t)f(t)dt.$$

Hence the expansion Theorem VI has to be modified as follows:

If F is of class C'', $R_0(F) = 0$, $R_1(F) = 0$, then

$$F(x) = \sum C_\nu\psi_\nu(x)$$

where

$$C_\nu = (F\psi_\nu).$$

Applying this theorem when $F = u$, $f = \lambda u$, $\lambda \neq 0$ we obtain: if u is continuous, then the statements, u is of class C'', $L(u) + \lambda u = 0$, $R_0(u) = 0$, $R_1(u) = 0$, $\lambda \neq 0$, imply and are implied by

$$u(x) = \lambda \int_a^b H(x, t)u(t)dt.$$

This establishes the equivalence of the boundary problem and the integral equation.

87. Flow of Heat in a Ring.—The deductions which we made in the previous problem for a straight bar hold for any linear conductor if we take for the independent variable x the length of arc from a fixed point O. The results also hold if the ring is closed and the two end points A and B coincide. When A and B coincide, the boundary conditions are modified as shown later.

FIG. 26.

Let us suppose $\Theta \equiv 0$, $A \equiv 0$, total length of ring $= 1$. Then the boundary problem becomes

$$(100) \qquad g\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(p\frac{\partial \theta}{\partial x}\right) + q\theta$$

$$\theta(0, t) = \theta(1, t);\ \theta_x(0, t) = \theta_x(1, t)$$

from the continuity of the temperature and of the flow of heat at A. Also

$$\theta(x, 0) = f(x), \text{ initial temperature.}$$

Put

$$\theta = u(x)\phi(t) \text{ in } (100).$$

We obtain

$$(101) \qquad \frac{d^2u}{dx^2} + (\lambda - b^2)u = 0$$

$$(102) \qquad u(0) = u(1),\ u'(0) = u'(1).$$

Case I.—$\lambda - b^2 > 0$, say $\lambda - b^2 = \mu^2$.

The general solution of (101) is

$$u = \alpha \cos \mu x + \beta \sin \mu x.$$

The boundary conditions (102) give the following two equations for the determination of α, β.

$$\begin{aligned} &\alpha(\cos\mu - 1) + \beta\sin\mu = 0 \\ -&\alpha\sin\mu + \beta(\cos\mu - 1) = 0. \end{aligned} \tag{103}$$

These two equations are compatible for values of α, β not both zero, if and only if

$$\begin{vmatrix} \cos\mu - 1 & \sin\mu \\ -\sin\mu & \cos\mu - 1 \end{vmatrix} = 0.$$

That is, if
$$2(1 - \cos\mu) = 0,$$
whence
$$\mu = 2n\pi.$$

For these values of μ, equations (103) are satisfied by all values of α and β. For other values of μ, the only solution is the trivial one $u \equiv 0$. The characteristic constants are then

$$\lambda_n = b^2 + 4n^2\pi^2,\ n = 1, 2, \ . \ . \ . \ ,$$

each of index 2.

The corresponding fundamental functions are

$$u = \alpha\cos 2n\pi x + \beta\sin 2n\pi x.$$

Case II.—$\lambda - b^2 = 0$. Then $\lambda = b^2$ is the only characteristic constant. u = constant is the only solution.

Case III.—$\lambda - b^2 < 0$. Then the only solution is the trivial one $u \equiv 0$. The characteristic constants of the problem are then $\lambda = b^2$, $\lambda_n = b^2 + 4n^2\pi^2$, $n = 1, 2, \ . \ . \ .$ with normalized fundamental functions

$$1,\ \sqrt{2}\cos 2n\pi x,\ \sqrt{2}\sin 2n\pi x$$

and, since $\int_0^1 \cos 2n\pi x \sin 2n\pi x\, dx = 0$, the complete normalized orthogonal system of fundamental functions is

$$1,\ \sqrt{2}\cos 2n\pi x,\ \sqrt{2}\sin 2n\pi x$$

with the characteristic constants

$$b^2,\ b^2 + 4n^2\pi^2,\ b^2 + 4n^2\pi^2.$$

This seems to contradict Theorem VII of §75, which stated that the index is always 1. The assumptions of this theorem were, however, that the boundary conditions were of the form

$$Au(a) + Bu'(a) = 0$$
$$Cu(b) + Du'(b) = 0.$$

The boundary conditions of the above problem are of a more general type:

$$A_0u(a) + B_0u'(a) + C_0u(b) + D_0u'(b) = 0$$
$$A_1u(a) + B_1u'(a) + C_1u(b) + D_1u'(b) = 0,$$

which explains the apparent contradiction.

If $b \neq 0$, (H_2) is satisfied and we can proceed to construct Green's function:

$$K(x, t) = \begin{cases} \dfrac{\cosh b(t - x - \frac{1}{2})}{2b \sinh \frac{b}{2}}, & x \leq t \\ \dfrac{\cosh b(x - t - \frac{1}{2})}{2b \sinh \frac{b}{2}}, & x \geq t \end{cases}$$

and the boundary problem is equivalent to the integral equation

$$u(x) = \lambda \int_a^b K(x, t)u(t)dt.$$

The expansion theorem can now be stated as follows: If F is of class C'', $F(0) = F(1)$ and $F'(0) = F'(1)$ then

$$F(x) = A_0 + \sum (A_n \cos 2n\pi x + B_n \sin 2n\pi x).$$

This is an ordinary Fourier series for the expansion of $F(x)$ for the interval $[01]$.

If $b = 0$, then (H_2) is not satisfied. Then, as in §86, we construct[1] a modified kernel:

$$H(x, t) = \begin{cases} \frac{1}{2}(t - x - \frac{1}{2})^2 + \frac{1}{24}, & x \leq t \\ \frac{1}{2}(x - t - \frac{1}{2})^2 + \frac{1}{24}, & t \leq x. \end{cases}$$

[1] See Kneser, "Integralgleichungen," §7.

88. Stationary Flow of Heat Produced by an Interior Source.—We have so far been considering the problem of the flow of heat under the assumption that no heat was produced in the interior of the conductor: $A \equiv 0$.

a) *Solution of the Problem of Stationary Flow of Heat.*—Let us now consider the case $A \not\equiv 0$. The equations of the problem now are

$$g\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(p\frac{\partial\theta}{\partial x}\right) + q\theta + A(x)$$

$$H_0\theta - \frac{\partial\theta}{\partial x} = 0 \text{ for } x = a,\ H_0 \geq 0$$

$$H_1\theta + \frac{\partial\theta}{\partial x} = 0 \text{ for } x = b,\ H_1 \geq 0.$$

Instead of the initial condition $\theta(x, 0) = f(x)$, we require that the flow of heat shall be stationary, that is, independent of the time, that is, $\theta = w(x)$ and, therefore, $\frac{\partial\theta}{\partial t} \equiv 0$.

Then the equations of the problem are

$$(104)\quad \frac{d}{dx}\left(p\frac{dw}{dx}\right) + qw + A(x) = 0 \quad \left[L(w) + A = 0\right]$$

$$(105)\quad \begin{aligned} R_0(w) &\equiv H_0w(a) - w'(a) = 0 \\ R_1(w) &\equiv H_1w(b) + w'(b) = 0. \end{aligned}$$

This is a non-homogeneous boundary problem. From Hilbert's third fundamental theorem we can write down at once the following:

If A is continuous, then the statements, w is of class C'', $L(w) + A = 0$, $R_0(w) = 0$, $R_1(w) = 0$, imply and are implied by

$$(106)\qquad w(x) = \int_a^b K(x, t)A(t)dt.$$

That is, the boundary problem (104) (105) has one and only one solution given by (106). Thus we have the theorem: *Every continuous source of heat $A(x)$ produces one and only one stationary flow of heat expressed by* (106).

b) Physical Interpretation of Green's Function.—From (106) Kneser obtains a physical interpretation for Green's function.

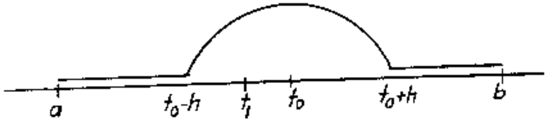


FIG. 27.

Take t_0 between a and b and let us suppose that

$$A(x)\begin{cases} = 0 \text{ for } a \leqq x \leqq t_0 - h,\ t_0 + h \leqq x \leqq b \\ \geqq 0 \text{ for } t_0 - h \leqq x \leqq t_0 + h. \end{cases}$$

Then, from the first mean value theorem for definite integrals, there exists a t_1, $t_0 - h \leqq t_1 \leqq t_0 + h$ such that

$$w(x,\,h) = \int_{t_0-h}^{t_0+h} K(x,\,t)A(t)dt = K(x,\,t_1)\int_{t_0-h}^{t_0+h} A(t)dt.$$

Now $A(x)dx$ is the quantity of heat produced in an element dx in unit of time, and hence $\int_a^b A(t)dt$ is the total amount of heat produced in the bar in unit time. This is called the *strength of source of heat.*

Let us suppose that the strength of the source of heat is 1:

$$\text{(107)} \qquad \int_{t_0-h}^{t_0+h} A(t)dt = 1.$$

Let now $h \to 0$ and let at the same time $A(x)$, which depends upon h, so vary that (107) remains satisfied. Then

$$K(x,\,t_0) = \lim_{h\to 0} w(x,\,h).$$

Thus we have the theorem: *In the general case, Green's function $K(x,\,t)$ represents the stationary temperature produced by a point source of strength unity placed at $x = t$.*

89. Direct Computation of the Characteristic Constants and Fundamental Functions.—In all of the examples in which we have been able actually to determine the characteristic constants and the fundamental functions, we have determined the latter by means of a boundary problem.

The question arises: How could we directly determine the characteristic constants, and the fundamental functions for an integral equation which does not correspond to a boundary problem? We might obtain the characteristic constants by the solution of the equation $D(\lambda) = 0$, and determine the index and the fundamental functions by means of the Fredholm minors. For some simple kernels this is easily done, but in the general case this method is hardly practicable.

Another direct method has been developed by Schmidt ("Diss.," pages 18 to 21) and Kneser ("Integralgleichungen," pages 190 to 197), at least for a symmetric kernel.

For simplicity, let us suppose that we knew *a priori* that all of the characteristic constants are positive and of index 1.

a) *Determination of* λ_1.—It was established in the general theory that if $0 < \lambda_1 < \lambda_2 < \ldots$ were the characteristic constants for $K(x, t)$ and $\psi_1(x), \psi_2(x), \ldots$ the corresponding complete normalized orthogonal system of fundamental functions, then the kernel $K_2(x, t)$ had the same fundamental functions and they belong to the characteristic constants $0 < \lambda_1^2 < \lambda_2^2 < \ldots$, for which we write $\mu_1 < \mu_2 < \ldots$. Now we have the following expansion for the logarithmic derivative of Fredholm's determinant $D(\lambda)$ derived from the kernel $K(x, t)$:

$$\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=0}^{\infty} U_{n+1}\lambda^n \qquad [(14), \S 54].$$

where $U_n = \int_a^b K_n(x, x)dx$, convergent for sufficiently small values of λ.

Hence if we call $D_2(\mu)$ the Fredholm determinant for $K_2(x, t)$, we obtain the corresponding expression

$$\frac{D'_2(\mu)}{D_2(\mu)} = -\sum_{n=0}^{\infty} U_{2n+2}\mu^n. \qquad (108)$$

Now $\dfrac{D'_2(\mu)}{D_2(\mu)}$ is a meromorphic function with simple poles, the poles being the roots of $D_2(\mu) = 0$. Arrange the roots $0 < \mu_1 < \mu_2 < \mu_3 < \ . \ . \ .$ in order of magnitude. Hence the circle of convergence of the expression (106) for $\dfrac{D'_2}{D_2}$ passes through μ_1 and hence the radius of convergence R is equal to μ_1.

Now there is a theorem[1] on power series to the effect that the radius of convergence R for the series $\sum A_n z^n$ is given by

$$\lim_{n\to\infty} \frac{A_n}{A_{n+1}} = R,$$

provided this ratio has a determinate limit as $n \to \infty$.

This theorem applied to the present problem gives

$$\lim_{n\to\infty} \frac{U_{2n}}{U_{2n+2}} = \mu_1,$$

provided the limit exists. That this ratio has a determinate limit follows from the monotony principle applied to the inequality

$$\frac{U_{2n-2}}{U_{2n}} \geqq \frac{U_{2n}}{U_{2n+2}} > 0$$

which we proved in §58.

b) Computation of the Fundamental Function $\psi_1(x)$.—From

$$K_{2n+2}(x, y) = \int_a^b K_2(x, t)K_{2n}(t, y)dt$$

we obtain

$$(109) \quad \mu_1^{n+1}K_{2n+2}(x, y) = \mu_1 \int_a^b K_2(x, t)\Big(\mu_1^n K_{2n}(t, y)\Big)dt.$$

Now Schmidt proves in his existence proof for the characteristic constants that

$$\lim_{n\to\infty} \mu_1^n K_{2n}(x, y) = f(x, y)$$

[1] HARKNESS and MORLEY, "Theory of Functions," §76.

a definite limit function which is approached uniformly with respect to x and y in the region R. Therefore, by passing to the limit in (109), we find

$$f(x, y) = \mu_1 \int_a^b K_2(x, t)f(t, y)dt.$$

Further Schmidt proves that $f(x, x) \not\equiv 0$. Hence, if for some quantity c we have $f(c, c) \neq 0$, then the function $f(x, c) = \varphi(x) \not\equiv 0$ and, therefore, $\varphi(x)$ is a fundamental function belonging to μ_1 since it satisfies the equation.

This function $\varphi(x)$, normalized in the usual way, gives us the fundamental function $\psi_1(x)$ belonging to λ_1.

c) Computation of the Other Characteristic Constants and Fundamental Functions.—The other characteristic constants and fundamental functions can be obtained successively as follows. Let

$$(110) \qquad K_2'(x, t) = K_2(x, t) - \frac{\psi_1(x)\psi_1(t)}{\mu_1}$$

be a new kernel, then we can show that its characteristic constants and fundamental functions are

$$\lambda_2^2 < \lambda_3^2 < \ . \ . \ . \ ; \psi_2(x), \psi_3(x), \ . \ . \ . \ .$$

For from (110) we obtain

$$(111) \quad \int_a^b K_2'(x, t)\psi_\nu(t)dt = \int_a^b K_2(x, t)\psi_\nu(t)dt + \frac{\psi_1(x)}{\mu_1}\delta_{1\nu},$$

where $\delta_{1\nu}$ is the Kronecker symbol. When $\nu = 1$, (111) gives

$$\int_a^b K_2'(x, t)\psi_1(t)dt = 0.$$

When $\nu \neq 1$, (111) gives

$$\psi_\nu(x) = \mu_\nu \int_a^b K_2'(x, t)\psi_\nu(t)dt,$$

which proves the above statement.

Applying now to the kernel $K_2'(x, t)$ the method described under a) and b), we obtain λ_2^2 and $\psi_2(x)$, and so on.

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